Riemannian Four-Manifolds with Nonnegative Curvature and Continuous Symmetry

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1 Introduction

A longstanding problem in Riemannian geometry is to determine which compact manifolds M^n admit Riemannian metrics g with positive sectional curvature. When $n = 2$ one can apply the Gauss-Bonnet theorem to conclude that a compact surface admitting a metric with positive curvature must be diffeomorphic to S^2 or RP^2. Compact positively curved manifolds (henceforth CPCM's) of dimension three were classified in [4]. They are diffeomorphic to spherical space forms. Four dimensional CPCM's are poorly understood. Synge's theorem [2] tells us that $\pi_1(M^4)$ is trivial or $\approx Z_2$ depending on whether M^4 is orientable or not, while in [6] it is shown that the sum of the Betti numbers of M^4 is less than a universal (astronomical) constant C_4. On the other hand, the only four dimensional CPCM's known are diffeomorphic to S^4, RP^4, or CP^2.

One of the goals of this thesis is to study a special class of four dimensional CPCM's: those which admit a continuous family of symmetries. A topological classification is given by

Theorem 1.0.1 *Let (M^4, g) be a connected orientable CPCM. Suppose (M^4, g) admits a nontrivial Killing vector field (i.e. an isometric R-action). Then M^4 is homeomorphic to S^4, or CP^2.*

The reasoning used in the proof of this theorem can be sharpened to prove

Theorem 1.0.2 *Let (M^4, g) be a one-connected compact nonnegatively curved Riemannian manifold. If (M^4, g) admits a nontrivial Killing vector field then M^4 is homeomorphic to S^4, CP^2, $S^2 \times S^2$, or $CP^2 \# \pm CP^2$.*

As in theorem 1.0.1, theorem 1.0.2 gives a complete classification since each of the manifolds listed in the conclusion of theorem 1.0.2 supports a metric g satisfying the hypotheses of the theorem (see [1]).

A compact nonnegatively curved manifold homeomorphic to $S^2 \times S^2$ or $CP^2 \# \pm CP^2$ which admits a nontrivial Killing vector field cannot have strictly positive curvature by theorem 1.0.1, so it is natural to investigate the distribution of flat (i.e. zero curvature) two-planes in such a Riemannian manifold. Some partial results in this direction can be found in sections 3.2 and 3.3.

In section 4 we consider compact nonnegatively curved Riemannian manifolds diffeomorphic to $S^2 \times S^2$ and $CP^2 \# -CP^2$ which admit nontrivial isometric actions of $SU(2)$. This symmetry assumption leads to a very strong structure theorems – essentially classification up to isometry – in most cases. Nonetheless, Kuranishi discovered some very interesting examples of nonnegatively curved metrics on $S^2 \times S^2$ which admit isometric $SO(3)$ actions.

Contents

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2 Preliminaries

This section gathers the geometric tools that will be used in the sequel. It may be skipped on a first reading without loss of continuity and used as a reference when needed.

In this section, (M^n, g) will be a complete connected Riemannian manifold with nonnegative curvature.

2.1 The Geometry of Manifolds with Nonnegative Curvature

Theorem 2.1.1 (Frankel) *Let $N_1^{k_1}, N_2^{k_2} \subseteq M^n$ be closed totally geodesic submanifolds with $k_1 + k_2 \geq n$, at least one of which is compact. Then either $N_1^{k_1} \cap N_2^{k_2} \neq \emptyset$ or there is a minimizing geodesic segment $\eta : [0, l] \longrightarrow M^n$ with*

1. *$\eta(0) = p_1 \in N_1^{k_1}$ and $\eta(l) = p_2 \in N_2^{k_2}$*
2. *$g(\eta'(0), T_{\eta(0)}N_1^{k_1}) = g(\eta'(l), T_{\eta(l)}N_2^{k_2}) = 0$*
3. *There is a parallel normal vector field $X : [0, l] \longrightarrow TM^n$ along η such that the sectional curvature is zero for each of the two-planes $span(\eta'(t), X(t)) \subseteq T_{\eta(t)}M^n$*

Proof: Suppose $N_1^{k_1} \cap N_2^{k_2} = \emptyset$. Then there is a shortest geodesic segment $\eta : [0, l] \longrightarrow M^n$ from $N_1^{k_1}$ to $N_2^{k_2}$. The first variation formula for arclength implies that $g(\eta'(0), T_{\eta(0)}N_1^{k_1}) = g(\eta'(l), T_{\eta(l)}N_2^{k_2}) = 0$, for otherwise one could shorten η while keeping its endpoints in $N_1^{k_1}$ and $N_2^{k_2}$.

Let $X : [0, l] \longrightarrow TM^n$ be a parallel normal vector field along η such that $X(0) \in T_{\eta(0)}N_1^{k_1}$ and $X(l) \in T_{\eta(l)}N_2^{k_2}$. Such a X exists: letting V be the vector space of parallel normal vector fields along η, $V_1 = \{X \in V | X(0) \in T_{\eta(0)}N_1^{k_1}\}$, and $V_2 = \{X \in V | X(l) \in T_{\eta(l)}N_2^{k_2}\}$, then $dim(V_1 \cap V_2) \geq dim V_1 + dim V_2 - dim V = k_1 + k_2 - (n-1) \geq 1$.

Consider the variation η_s of η given by $\eta_s(t) = exp(sX(t)), s \in R, t \in [0, l]$. Then

$$\frac{d^2}{ds^2}(length(\eta_s))|_{s=0} = \int_0^l g(R(X(t), \eta'(t))X(t), \eta'(t))dt \leq 0$$

where $R(\cdot,\cdot)$ is the curvature tensor of M^n. Since η is a shortest path from $N_1^{k_1}$ to $N_2^{k_2}$, we conclude that

$$\frac{d^2}{ds^2}(length(\eta_s))|_{s=0} = 0$$

forcing $g(R(X(t),\eta'(t))X(t),\eta'(t)) = 0$ for all $t \in [0,l]$. $\square$

In the second case of theorem 2.1.1 one can prove more with the following lemma. It will be used in sections 3.2, 3.3, and 4.

Lemma 2.1.2 *Let $\gamma_i : [0,S] \longrightarrow M^n, i = 1,2$ and $\eta_0 : [0,l] \longrightarrow M^n$ be geodesic segments. Suppose $\eta_0(0) = \gamma_1(0), \eta_0(l) = \gamma_2(0), g(\eta_0'(0),\gamma_1'(0)) = g(\eta_0'(l),\gamma_2'(0)) = 0$, and $X : [0,l] \longrightarrow TM^n$ is a parallel normal vector field along η_0 with $X(0) = \gamma_1'(0), X(l) = \gamma_2'(0)$. Define $\eta : [0,S] \times [0,l] \longrightarrow M^n$ by $\eta(s,t) = exp(sX(t))$ and $\eta_s : [0,l] \longrightarrow M^n$ by $\eta_s = \eta \circ (s, id)$. Then either $length(\eta_s) < length(\eta_0) = l$ for some $s \in [0,S]$, or*

1. *$length(\eta_s) = length(\eta_0)$ for every $s \in [0,S]$.*
2. *η is an isometric immersion.*
3. *$\eta_*(\frac{\partial}{\partial t})$ is parallel along each $\eta|_{[0,S]\times\{t\}}$, where $\frac{\partial}{\partial t}$ is the second coordinate vector field on $[0,S] \times [0,l]$.*

Proof: Assume that $length(\eta_s) \geq length(\eta_0)$ for all $s \in [0,S]$. Let $[0,S_1] \subseteq [0,S]$ be the largest subinterval of $[0,S]$ for which

1. $length(\eta_s) = length(\eta_0)$, $s \in [0,S_1]$
2. $\eta|_{[0,S_1]\times[0,l]}$ is an isometric immersion,
3. $\eta_*(\frac{\partial}{\partial t})$ is parallel along $\eta|_{[0,S_1]\times\{t\}}$ for all $t \in [0,l]$.

We will assume that $S_1 < S$, and prove the lemma by contradiction. Consider the Jacobi field $J_t = \eta_*(\frac{\partial}{\partial t})$ along the geodesic $\eta|_{[0,S_1]\times\{t\}}$. This is a parallel Jacobi field on the segment $\eta|_{[0,S_1]\times\{t\}}$, so $(\nabla_{\frac{\partial}{\partial s}} J_t)(S_1,t) = 0$. Hence by theorem 1.29 [2], $|J_t(s)| \leq |J_t(S_1)|$ for $s \in [S_1, S_1+\epsilon]$, with equality holding for every $s \in [S_1, S_1+\epsilon]$ only when J_t is parallel on the interval $[S_1, S_1+\epsilon]$,

and $\epsilon > 0$ depends only on curvature bounds for M^n in a neighborhood of $\eta([0,S] \times [0,l])$. For $s \in [S_1, S_1 + \epsilon]$,

$$length(\eta_s) = \int_0^l |\eta_s'(t)|dt = \int_0^l |J_t(s)|dt$$

$$\leq \int_0^l |J_t(S_1)|dt = length(\eta_{S_1}) = length(\eta_0)$$

Since $length(\eta_s) \geq length(\eta_0)$ we conclude that J_t is parallel on $[0, S_1 + \epsilon]$ for every $t \in [0,l]$. This shows that $\eta|_{[0,S_1+\epsilon]\times[0,l]}$ is an isometric immersion and $\eta_*(\frac{\partial}{\partial t})$ is parallel along $\eta|_{[0,S_1]\times\{t\}}$ for each $t \in [0,l]$. This contradicts the definition of $S_1 \in (0,S)$, proving lemma 2.1.2. $\square$

Corollary 2.1.3 *If γ_i, η_0, η, etc. are as in lemma 2.1.2 and $dist(\gamma_1, \gamma_2) = length(\eta_0)$, then η is a totally geodesic isometric immersion.*

Proof: By lemma 2.1.2, η is an isometric immersion and $\eta_*(\frac{\partial}{\partial t})$ is parallel along $\eta|_{[0,S]\times\{t\}}$ for each $t \in [0,l]$, so if II is the second fundamental form of the immersion η, then

$$II(\eta_*(\frac{\partial}{\partial s}), \eta_*(\frac{\partial}{\partial s})) = II(\eta_*(\frac{\partial}{\partial s}), \eta_*(\frac{\partial}{\partial t})) = 0$$

The curves η_s are all minimizing paths from γ_1 to γ_2, so $II(\eta_*(\frac{\partial}{\partial t}), \eta_*(\frac{\partial}{\partial t})) = 0$ as well. Hence $II \equiv 0$, and η is totally geodesic. $\square$

We now turn to the geometry of geodesic triangles in nonnegatively curved manifolds.

Definition 2.1.4 *If $p, q \in M^n$, and $dist(p,q) = l$, let*

$$Min_p(q) = \{\gamma : [0,l] \longrightarrow M^n | \gamma(0) = p, \gamma(l) = q, and |\gamma'| \equiv 1\}$$

$$= \{minimizing\ paths\ from\ p\ to\ q\}$$

$$Dir_p(q) = \{initial\ directions\ of\ \gamma \in Min_p(q)\}$$

$$= \{\gamma'(0) | \gamma \in Min_p(q)\}$$

$$\angle_p(q,r) = \angle_p(Dir_p(q), Dir_p(r))$$

Theorem 2.1.5 (Toponogov) *1. If $p_1, p_2, p_3 \in M^n$, then $\angle_{p_1}(p_2, p_3) + \angle_{p_2}(p_1, p_2) + \angle_{p_3}(p_3, p_1) \geq \pi$.*

2. If equality holds in 1, and $\gamma_2 \in Min_{p_1}(p_2), \gamma_3 \in Min_{p_1}(p_3)$ with $\angle(\gamma_2'(0), \gamma_3'(0)) = \angle_{p_1}(p_2, p_3)$, then there is an embedded, flat, totally geodesic triangular surface $\Delta^2 \subseteq M^n$ with geodesic edges γ_2, γ_3, and $\eta \in Min_{p_2}(p_3)$.

Proof: Let $l_{ij} = dist(p_i, p_j), 1 \leq i, j \leq 3, i \neq j$, and construct a triangle $\hat{\Delta} \subseteq E^2$ with vertices $\hat{p}_i \in E^2$ and $dist(\hat{p}_i, \hat{p}_j) = l_{ij}$.

Given $\gamma_2 \in Min_{p_1}(p_2), \gamma_3 \in Min_{p_1}(p_3)$ we may apply Toponogov's theorem [2] to conclude that $\angle_{p_1}(\gamma_2'(0), \gamma_3'(0)) \geq \angle(\overline{\hat{p}_1\hat{p}_2}, \overline{\hat{p}_1\hat{p}_3})$. Hence $\angle_{p_1}(p_2, p_3) = \angle(Dir_{p_1}(p_2), Dir_{p_1}(p_3)) \geq \angle(\overline{\hat{p}_1\hat{p}_2}, \overline{\hat{p}_1\hat{p}_3})$. Cyclic repetition of this reasoning gives

$$\begin{aligned} &\angle_{p_1}(p_2, p_3) + \angle_{p_2}(p_1, p_2) + \angle_{p_3}(p_3, p_1) \\ &\geq \angle(\overline{\hat{p}_1\hat{p}_2}, \overline{\hat{p}_1\hat{p}_3}) + \angle(\overline{\hat{p}_3\hat{p}_1}, \overline{\hat{p}_3\hat{p}_2}) + \angle(\overline{\hat{p}_2\hat{p}_3}, \overline{\hat{p}_2\hat{p}_1}) \\ &= \pi \end{aligned}$$

The second part of the theorem follows from the discussion of the equality case of Toponogov's theorem in [2]. □

2.2 The Geometry of Nonnegatively Curved Manifolds with Symmetry

In this section we fix a compact Lie group G with Lie algebra $L(G)$ and an isometric action $\Phi : G \times M^n \longrightarrow M^n$. $G(p)$, for $p \in M^n$ will denote the G orbit $\{g \cdot p | g \in G\}$ of p, and G_p will denote the isotropy subgroup $\{g \in G | g \cdot p = p\}$. The notation for the induced action of G on TM^n will be similar, except that elements of TM^n will be used in place of elements of M^n.

Definition 2.2.1 *A geodesic segment $\gamma : [0, l] \longrightarrow M^n$ is a G-geodesic segment if $g(\gamma'(t), T_{\gamma(t)}G(\gamma(t))) = 0$ for all $t \in [0, l]$, and G-minimizing if $length(\gamma) = dist(G(\gamma(0)), G(\gamma(l)))$.*

Lemma 2.2.2 *1. (Clairaut) If $\gamma : [0, l] \longrightarrow M^n$ is a geodesic with $g(\gamma'(0), T_{\gamma(0)}G(\gamma(0))) = 0$ then γ is a G-geodesic segment.*

2. A G-minimizing segment is a G-geodesic segment.

3. *If $\gamma : [0,l] \longrightarrow M^n$ is G-minimizing, then $G_{\gamma(\cdot)}$ is constant on $(0,l)$, and $G_{\gamma(t)} \equiv G_\gamma \stackrel{def}{=} \{g \in G | \, g\big|_{Im(\gamma)} = id_{Im\gamma}\}$ for $t \in (0,l)$.*

Proof: 1 and 2 follow immediately from the first variation formula for arclength.

To see 3, pick $t \in (0,l)$, and suppose $G_{\gamma(t)} \neq G_\gamma$. Then there exists $g \in G_{\gamma(t)}$ such that $g \cdot \gamma'(t) \neq \gamma'(t)$, for otherwise $G_{\gamma(t)} \subseteq G_\gamma$. The segments $\gamma\big|_{[0,t]}$ and $g \cdot \gamma\big|_{[t,l]}$ fit together to give a nonsmooth path η from $\gamma(0) \in G(\gamma(0))$ to $g \cdot \gamma(l) \in G(\gamma(l))$ and $length(\eta) = length(\gamma)$. This contradicts the minimality of γ. $\square$

When G acts freely on M^n or more generally when the G action has only on orbit type, then M^n/G inherits a canonical smooth quotient Riemannian metric, the submersion metric. The submersion formula of O'Neill guarantees that this metric has nonnegative curvature, so one may apply the results in section 2.1. Proposition 2.2.7 below and the discussion preceeding it provide a partial analog for the second variation formula in M^n/G when the G action has more than one orbit type, and the quotient metric isn't smooth.

The first step is the formulation of G-parallelism given in definition 2.2.5.

Definition 2.2.3 *If $\eta : [0,l] \longrightarrow M^n$ is a G-geodesic, then the vertical bundle $Vert^G_\eta$ along η has as fiber over $\eta(t)$,*

$$Vert^G_\eta(t) = span((T_{\eta(t)}G(\eta(t))), \{\frac{d}{ds}(expsX) \cdot \eta'(t)\big|_{s=0} | \, X \in L(G_{\eta(t)})\})$$

where $L(G_{\eta(t)})$ is the Lie algebra of $G_{\eta(t)}$.

If $G(\eta(t))$ is a principal orbit then $Vert^G_\eta$ will have fiber $T_{\eta(t)}G(\eta(t))$ over $\eta(t)$ since $G_{\eta(t)}$ acts trivially on $\nu_{\eta(t)}G(\eta(t))$. Hence $Vert^G_\eta$ will actually be a smooth vector bundle near $\eta(t)$. Alternatively, note that the quotient $M^n \longrightarrow M^n/G$ defines a Riemannian submersion near $\eta(t)$ and the vertical bundle is just the restriction of the usual vertical bundle to η.

Lemma 2.2.4 *$Vert^G_\eta$ is a smooth vector bundle along η.*

Proof: Fix $t_0 \in [0,l]$, and consider $Vert^G_\eta$ near $\eta(t_0)$.

First suppose $\eta(t_0) \in Fix(G, M^n) = \{p \in M^n |\ g \cdot p = p,\ \forall g \in G\}$. Then the G action is locally smoothly equivalent via the exponential map $T_{\eta(t_0)}M^n \longrightarrow M^n$ to the orthogonal action of G on $T_{\eta(t_0)}M^n$. The exponential local equivalence carries a piece of η to a line segment in $T_{\eta(t_0)}M^n$, so the smoothness of $Vert_\eta^G$ near $\eta(t_0)$ is immediate.

In the general case, let $H = G_{\eta(t_0)}$, and let $W \subseteq L(G)$ be a complement to $L(H)$ in $L(G)$. Consider the bundles $Vert_\eta^H$ and $\tilde{W}$ near $\eta(t_0)$, where the fiber of $\tilde{W}$ at $\eta(t)$ is $\{\frac{d}{ds}(expsX) \cdot \eta(t)\big|_{s=0} |\ X \in W\}$. Both bundles are smooth, and their fibers are linearly independent at $\eta(t_0)$ since $G_{\eta(t_0)}$ preserves the splitting

$$T_{\eta(t_0)}M^n = T_{\eta(t_0)}G(\eta(t_0)) \oplus \nu_{\eta(t_0)}G(\eta(t_0)).$$

Claim: $Vert_\eta^G = Vert_\eta^H \oplus \tilde{W}$ near t_0.

Proof of claim: The fiber of $Vert_\eta^G$ at $\eta(t)$ is spanned by $T_{\eta(t)}G(\eta(t))$ and $\{\frac{d}{ds}(expsX) \cdot \eta'(t)\big|_{s=0} | X \in L(G_{\eta(t)})\}$. Assuming $\eta(t)$ is within a normal tubular neighborhood of $G(\eta(t_0))$, then $G_{\eta(t)} \subseteq G_{\eta(t_0)}$. In fact, if $t \neq t_0$ then $G_{\eta(t)} = \{g \in G_{\eta(t_0)} |\ g \cdot \eta'(t_0) = \eta'(t_0)\} = H_{\eta'(t_0)}$. Therefore if $t \neq t_0$ and $\eta(t)$ is close to $\eta(t_0)$, then $\{\frac{d}{ds}(expsX) \cdot \eta'(t)\big|_{s=0} |\ X \in L(G_{\eta(t)})\} = \{0\}$, and $Vert_\eta^G(t) = T_{\eta(t)}G(\eta(t))$. Since $(Vert_\eta^H \oplus \tilde{W})(t)$ spans $T_{\eta(t)}G(\eta(t))$ for $\eta(t)$ close to $\eta(t_0)$, the proof is complete. $\square$

Definition 2.2.5 *If $\eta : [0,l] \longrightarrow M^n$ is a G-geodesic segment and $X : [0,l] \longrightarrow TM^n$ is a vector field along η, then X is G-parallel if X is a parallel section of the smooth bundle $Hor_\eta^G = (Vert_\eta^G)^\perp$ along η. In other words $g(X(t), Vert_\eta^G(t)) = 0$ and $\nabla_{\eta(t)}X(t) \in Vert_\eta^G(t)$ for every $t \in [0,l]$.*

Remark 2.2.6 *Since $Vert_\eta^G$ is smooth, one can extend any $X(t_0) \in Hor_\eta^G(t_0) = (Vert_\eta^G(t_0))^\perp$ to a G-parallel section in the usual way. This is the motivation for lemma 2.2.4.*

Proposition 2.2.7 *Let $\gamma_1, \gamma_2 : [0,S] \longrightarrow M^n$, $\eta_0 : [0,l] \longrightarrow M^n$ be G-geodesic segments with $\eta_0((0,l)) \subseteq (M^n)^\circ$ =the union of the principal G-orbits. Suppose $\eta_0(0) = \gamma_1(0)$, $\eta_0(l) = \gamma_2(0)$, $g(\eta_0'(0), \gamma_1'(0)) = g(\eta_0'(l), \gamma_2'(0)) = 0$, and $X : [0,l] \longrightarrow TM^n$ is a G-parallel vector field along η_0 with $X(0) = \gamma_1'(0)$, $X(l) = \gamma_2'(0)$. Then either $dist(G(\gamma_1(s)), G(\gamma_2(s))) < length(\eta_0) = l$ for some $s \in [0,S]$, or X is parallel along η_0 in the ordinary sense of the word.*

Corollary 2.2.8 *Let γ_1, γ_2, η_0, and X be as in 2.2.7, and suppose $dist(G(\gamma_1(s)), G(\gamma_2(s))) \geq length(\eta_0) = l$ for every $s \in [0, S]$. Define $\eta : [0, S] \times [0, l] \longrightarrow M^n$ by $\eta(s,t) = exp(sX(t))$. Then η is a totally geodesic isometric immersion of the Euclidean rectangle $[0, S] \times [0, l]$. Moreover η is orthogonal to the G-action: $g(\eta_*(Y), T_{\eta(s,t)}G(\eta(s,t))) = 0$ for every $Y \in T_{(s,t)}([0, S] \times [0, l])$ and every $(s,t) \in [0, S] \times [0, l]$.*

Proof of corollary 2.2.8: By proposition 2.2.7, X is a parallel vector field along η. Now lemma 2.1.2 and corollary 2.1.3 can be applied, showing that η is a totally geodesic isometric immersion. To see that η is orthogonal to the G action, note that each segment $\eta_s = \eta \circ (s, id_{[0,l]}) : [0, l] \longrightarrow M^n$ is G-minimizing, so $g(\eta_s'(t), T_{\eta_s(t)}G(\eta_s(t))) = 0$ for all $t \in [0, l]$, while the segments $\eta \circ (id_{[0,S]}, t) : [0, S] \longrightarrow M^n$ are G-geodesic segments by part 1 of lemma 2.2.2. $\square$

Proof of 2.2.7: The idea is to try to apply the ordinary second variation formula in the orbit space M^n/G, which has nonnegative curvature in its smooth part by O'Neill's curvature formula for Riemannian submersions[2].

Let η and η_s be defined as in lemma 2.1.2. Picking $\delta \in (0, l/2)$, $s \in [0, S]$, we have

$$dist(G(\gamma_1(s)), G(\gamma_2(s))) = dist(G(\eta(s,0)), G(\eta(s,l)))$$

$$\leq dist(G(\eta(s,0)), G(\eta(s,\delta))) + dist(G(\eta(s,\delta)), G(\eta(s,l-\delta))) + dist(G(\eta(s,l-\delta)), G(\eta(s,l)))$$

$$\leq dist(G(\eta(s,\delta)), G(\eta(s,l-\delta))) + dist(\eta(s,0), \eta(s,\delta)) + dist(\eta(s,l-\delta), \eta(s,l))$$

Looking at the last two terms we have

$$dist(\eta(s,0), \eta(s,\delta)) \leq length(\eta_s\big|_{[0,\delta]})$$

$$\leq length(\eta_0\big|_{[0,\delta]}) + \delta C_1 s^2$$

where C_1 is a constant independent of δ and s. Likewise

$$dist(\eta(s,l-\delta), \eta(s,l))$$

$$\leq length(\eta_0\big|_{[l-\delta,l]}) + \delta C_2 s^2$$

for some C_2 independent of δ and s.

The first term can be estimated by working in the smooth, nonnegatively curved Riemannian manifold $(M^n)^s/G$. Define $\bar{\eta}_s : [\delta, l-\delta] \longrightarrow M^n/G$ by $\bar{\eta}_s = \pi \circ \eta_s\big|_{[\delta,l-\delta]}$ where $\pi : M^n \longrightarrow M^n/G$ is the quotient map. For small s, $\bar{\eta}_s$ defines a a variation of $\bar{\eta}_0$ in $(M^n)^s/G$ =the smooth part of M^n/G, with parallel variation vector field $\bar{X} = \pi_* \circ X$. Hence the second variation formula gives

$$\frac{d^2}{ds^2} length(\eta_s)\Big|_{s=0} = \int_\delta^{l-\delta} g(\bar{R}(\bar{\eta}_0'(t), \bar{X}(t))\bar{\eta}_0'(t), \bar{X}(t))dt$$

$$\stackrel{def}{=} I_\delta$$

where $\bar{R}$ is the curvature tensor of $(M^n)^s/G$. Combining these estimates we get

$$dist(G(\gamma_1(s)), G(\gamma_2(s)))$$
$$\leq length(\eta_0) + (\delta(C_1 + C_2) + I_\delta + error(s))s^2$$

where $\lim_{s\to 0}(error(s)) = 0$.

Since I_δ is monotone in δ, either $dist(G(\gamma_1(s), G(\gamma_2(s))) < length(\eta_0) = l$ for some $s \in [0, S]$ or $I_\delta \equiv 0$. In the latter case $g(\bar{R}(\bar{\eta}_0'(t), X(t))\bar{\eta}_0'(t), \bar{X}(t)) = 0$ for every $t \in [0, l]$, implying via O'Neill's formula that the horizontal lift of $\bar{X}$, i.e. X, is parallel along η_0. $\square$.

2.3 A Lemma

Lemma 2.3.1 *Let $\rho : G \longrightarrow O(n)$ be an orthogonal representation of a compact Lie group G, and suppose $v, w \in E^n \setminus \{0\}$. If ρ has no trivial summands then $\sup_{g\in G}\langle g \cdot v, w\rangle \geq 0$ with equality only if ρ is reducible and $G(v)$, $G(w)$ lie in mutually orthogonal direct summands of ρ.*

Proof: If $x \in E^n$, then $Average_{g\in G}(g\cdot x) \in Fix(G, E^n) = \{0\}$, so $Average_{g\in G}(g\cdot x) = 0$. Therefore $\sup_{g\in G}\langle g \cdot v, w\rangle \geq Average_{g\in G}\langle g \cdot v, w\rangle = \langle Average_{g\in G}(g \cdot v, w\rangle = \langle 0, w\rangle = 0$. Equality holds only if $\langle g \cdot v, w\rangle = 0$ for every $g \in G$, which gives $\langle span(G(v)), span(G(w))\rangle = 0$. $\square$

3 The Proofs of Theorems 1.0.1 and 1.0.2

The main goal in this section is to prove theorems 1.0.1 and 1.0.2. Section 3.1 reduces these two theorems to the analysis of three cases, which are dealt with in sections 3.2, 3.3, and 3.4. In addition to pieces of the proof of 1.0.1 and 1.0.2, sections 3.2 and 3.3 contain some further information on the structure of certain manifolds satisfying the hypotheses of 1.0.2. See theorem 3.2.2 and theorem 3.3.6.

3.1 Reduction of 1.0.1 and 1.0.2

Theorems 1.0.1 and 1.0.2 are corollaries of

Theorem 3.1.1 *Let (M^4, g) be a four dimensional, connected, compact, nonnegatively curved manifold admitting a nontrivial Killing vector field. Then*

1. $\chi(M^4) \leq 4$
2. $\chi(M^4) = 4$ *only if* (M^4, g) *has some flat two-planes, i.e. two-planes with zero sectional curvature.*

Proof of theorems 1.0.1 and 1.0.2 using theorem 3.1.1:

The hypotheses of theorem 1.0.1 together with Synge's theorem [2] imply that $\pi_1(M^4) = \{1\}$. Hence $b_1(M^4) = rank(H_1(M^4, Z)) = 0$ and by Poincaré duality $b_3(M^4) = 0$. Now

$$\chi(M^4) = \sum_{i=0}^{4}(-1)^i b_i(M^4)$$

$$= b_0(M^4) + b_2(M^4) + b_4(M^4)$$

$$= 2 + b_2(M^4)$$

Applying theorem 3.1.1, we get $b_2(M^4) = \chi(M^4) - 2 \leq 1$. Freedman's topological classification of simply connected four-manifolds [3] now gives $M^4 \approx S^4$, CP^2.

The proof of theorem 1.0.2 from theorem 3.1.1 is similar except that one has to check that all the possibilities for an intersection form for M^4 are realized by those of the listed manifolds when $rank(H_2(M^4, Z)) \leq 2$. $\square$

Proof of theorem 3.1.1: To prove theorem 3.1.1 we will assume that $\chi(M^4) \geq 4$ and will show that $\chi(M^4) = 4$ and the sectional curvature of (M^4, g) isn't strictly positive.

The hypotheses imply that the isometry group of (M^4, g) is a positive dimensional compact Lie group G. We pick a circle subgroup $S^1 \subseteq G$ and we will use the corresponding effective isometric S^1 action $S^1 \times M^4 \longrightarrow M^4$.

Let $Fix(S^1, M^4) = \{p \in M^4 | S^1(p) = p\}$ = *the fixed point set of* S^1 *in* M^4. Smith theory and theorem [5] imply that

1. $\chi(Fix(S^1, M^4)) = \chi(M^4)$.
2. $Fix(S^1, M^4)$ is a disjoint union of (a finite number of) totally geodesic submanifolds, each of which has even codimension in M^4.

In our case, $Fix(S^1, M^4)$ is a disjoint union of isolated points and totally geodesic surfaces with $\chi(Fix(S^1, M^4)) = \chi(M^4) \geq 4$. This implies that $Fix(S^1, M^4)$ contains one of the following subsets:

1. Two disjoint totally geodesic surfaces.
2. A totally geodesic surface and two isolated points.
3. Four isolated points.

In sections 3.2, 3.3, and 3.4 respectively we show that cases 1, 2, 3 each lead to the conclusion that $\chi(M^4) = \chi(Fix(S^1, M^4)) = 4$ and the sectional curvature of M^4 isn't strictly positive. This proves theorem 3.1.1. □

3.2 $Fix(S^1, M^4)$ Contains Two Totally Geodesic Surfaces

In this section we study the geometry of a four dimensional nonnegatively curved manifold (M^4, g) and an effective isometric S^1 action $S^1 \times M^4 \longrightarrow M^4$ whose fixed point set $Fix(S^1, M^4)$ contains a pair of disjoint totally geodesic surfaces, N_1^2 and N_2^2. In corollary 3.2.3 we show that $\chi(M^4) = \chi(Fix(S^1, M^4)) \leq 4$, and in observation 3.2.1 we point out that the sectional curvature of (M^4, g) isn't strictly positive.

Observation 3.2.1 *Theorem 2.1.1 implies that M^4 doesn't have strictly positive sectional curvature.*

The main goal of this section is to prove

Theorem 3.2.2 *1. $Fix(S^1, M^4) = N_1^2 \amalg N_2^2$.*

2. S^1 acts freely on $M^4 \setminus (N_1^2 \amalg N_2^2)$.

3. $(M^4 \setminus (N_1^2 \amalg N_2^2))/S^1$ is isometric to $N_1^2 \times (0, l)$, where $l = dist(N_1^2, N_2^2)$.

For the proof of theorem 3.2.2 see lemmas 3.2.5 and 3.2.7.

Part 3 of theorem 3.2.2 is really a splitting theorem for M^4/S^1. We avoid this formulation since it would involve the unnecessary complication of working in the (a priori) nonsmooth orbit space M^4/S^1.

Corollary 3.2.3 $\chi(M^4) = \chi(Fix(S^1, M^4)) \leq 4$

Proof: By theorem 3.2.2 we have $\chi(Fix(S^1, M^4)) = \chi(N_1^2) + \chi(N_2^2) \leq 2 + 2 = 4$ $\square$

Lemma 3.2.4 *Suppose $p \in N_1^2$ satisfies $dist(p, N_2^2) = dist(N_1^2, N_2^2) = l$, and $\eta_0 : [0, l] \longrightarrow M^4$ is a geodesic segment with $\eta_0(0) = p$, $\eta_0(l) = q \in N_2^2$, and $length(\eta_0) = l$. Then*

1. Any $X(0) \in T_pN_1^2$ extends to an S^1-parallel (see definitions 2.2.1 and 2.2.3), parallel, Jacobi field $X : [0, l] \longrightarrow TM^4$ along η_0 corresponding to a variation through geodesic segments normal to N_1^2.

2. $Dist(exp(sX(0)), N_2^2) = l$ for every $s \geq 0$.

3. $Dist(p', N_2^2) = l$ for every $p' \in N_1^2$.

Proof: We will assume the reader is familiar with section 2.2. Since $length(\eta_0) = dist(N_1^2, N_2^2) = dist(p, q) = dist(G(p), G(q))$, we have $g(\eta_0'(0), T_pN_1^2) = g(\eta_0'(l), T_qN_2^2) = 0$ and η_0 is an S^1-minimizing geodesic segment (see definition 2.2.1). Lemma 2.2.2 implies that $S^1_{\eta_0(t)}$ is trivial for $t \in (0, l)$. The representation of S^1 on T_pM^4 preserves $T_pN_1^2$, and so $S^1(\eta_0'(0)) \subseteq \nu_pN_1^2 \stackrel{def}{=}$ the normal space of N_1^2 at p. Therefore $X(0) \in T_pN_1^2$ extends to an S^1-parallel vector field $X : [0, l] \longrightarrow TM^4$ along η_0 by remark 2.2.6. Note that since $X(l) \in (Vert^{S^1}_{\eta_0}(l))^\perp$ and $g(X(l), \eta_0'(l)) = 0$, we have $X(l) \in T_{\eta_0(l)}N_2^2$.

Define $\gamma_1 : [0, \infty) \longrightarrow M^4$, $\gamma_2 : [0, \infty) \longrightarrow M^4$ by $\gamma_1(s) = exp(sX(0))$ and $\gamma_2 = exp(sX(l))$. Then $\gamma_1([0, \infty)) \subseteq N_1^2$ and $\gamma_2([0, \infty)) \subseteq N_2^2$ since N_1^2

and N_2^2 are totally geodesic. Applying corollary 2.2.8, we get parts 1 and 2 of the lemma. To get part 3 of the lemma, pick $X(0) \in T_pN_1^2$ so that $exp(sX(0)) = p'$ for some $s \in [0, \infty)$ and then use part 2. $\square$

Lemma 3.2.5 *1. $exp\big|_{D_l\nu N_1^2} : D_l\nu N_1^2 \longrightarrow M^4$ is a diffeomorphism onto $M^4 \setminus N_2^2$*

2. S^1 acts freely on $M^4 \setminus (N_1^2 \amalg N_2^2)$

Proof: It follows from the previous lemma that if $p \in N_1^2$ is arbitrary and $\eta_0 : [0, l] \longrightarrow M^4$ is a geodesic segment with $\eta_0'(0) \in \nu N_1^2$, then η_0 is an S^1-minimizing geodesic segment from $\eta_0(0) \in N_1^2$ to $\eta_0(l) \in N_2^2$, and in fact a minimizing segment from $\eta_0(0)$ to N_2^2 or from $\eta_0(l)$ to N_1^2. Therefore the normal injectivity radius of N_1^2 is l, $exp\big|_{D_l\nu N_1^2}$ is a diffeomorphism onto its image, and S^1 acts freely on $exp(D_l\nu N_1^2) \setminus N_1^2$.

Any geodesic extension of η_0 will fail to be minimizing by part 3 of lemma 2.2.2, so $dist(q, N_1^2) \leq l$ for every $q \in M^4$ and $dist(q, M^4) = l$ only when $q \in N_2^2$. This proves that $exp(D_l\nu N_1^2) = M^4 \setminus N_2^2$ and part 2 of the lemma. $\square$

Let $\pi : D_l\nu N_1^2 \longrightarrow N_1^2$ be the bundle projection, and define $P : M^4 \setminus N_2^2 \longrightarrow N_1^2$ by $P = \pi \circ (exp\big|_{D_l\nu N_1^2})^{-1}$.

Lemma 3.2.6 *P is an S^1-equivariant Riemannian submersion.*

Proof: The S^1 equivariance of P follows from that of π and $exp\big|_{D_l\nu N_1^2}$.

The fiber $P^{-1}(p)$, for $p \in N_1^2$, is just $exp(D_l\nu_p N_1^2)$. If $q \in P^{-1}(p) \setminus \{p\}$, and $q = \eta_0(t)$ for some $t \in (0, l)$, and some geodesic segment $\eta_0 : [0, l] \longrightarrow M^4$ with $\eta_0'(0) \in \nu_p N_1^2$, then $T_q(P^{-1}(p))$ is spanned by $\eta_0'(t)$ and $T_qS^1(q)$. Hence $\nu_q P^{-1}(p)$ is spanned by the various $X(t) \in T_qM^4$ as X runs over all the Jacobi vector fields produced by lemma 3.2.4. Since $P_*X(t) = X(0)$ for such a Jacobi field, $P_*\big|_{\nu_q P^{-1}(p)} : \nu_q P^{-1}(p) \longrightarrow T_pN_1^2$ is an orthogonal linear transformation. Therefore P is a Riemannian submersion. $\square$

Define $d : M^4 \setminus (N_1^2 \amalg N_2^2) \longrightarrow (0, l)$ by $d(q) = dist(q, N_1^2)$. This is a smooth Riemannian submersion since the normal injectivity radius of N_1^2 is l.

Lemma 3.2.7 *1. The map $P \times d : M^4 \setminus (N_1^2 \amalg N_2^2) \longrightarrow N_1^2 \times (0,l)$ is a Riemannian submersion if $N_1^2 \times (0,l)$ is given the product metric.*

2. $P \times d$ induces an isometry $\overline{P \times d} : (M^4 \setminus (N_1^2 \amalg N_2^2))/S^1 \longrightarrow N_1^2 \times (0,l)$.

Proof: To see 1, pick $q \in M^4 \setminus (N_1^2 \amalg N_2^2)$ and note that the horizontal subspaces of the Riemannian submersions P and d, i.e. $(kerP_{*q})^\perp$ and $(kerd_{*q})^\perp$ respectively, are orthogonal subspaces of T_qM^4.

2 follows immmediately from the S^1-equivariance (or S^1- invariance) of $P \times d$. $\square$

3.3 $Fix(S^1, M^4)$ Contains a Totally Geodesic Surface and Two Isolated Points

In this section we consider a four-dimensional, compact, connected, nonnegatively curved Riemannian manifold (M^4, g) and an isometric S^1-action on (M^4, g) whose fixed point set contains at least two isolated points, p_1, p_2, and a totally geodesic surface N^2. We will show that the sectional curvature of (M^4, g) isn't strictly positive and that $Fix(S^1, M^4) = \{p_1, p_2\} \cup N^2$, which implies that $\chi(M^4) \leq 4$. In addition, we show that a dense open subset of M^4 admits an S^1-equivariant Riemannian submersion to an open rectangle $(0, l_1) \times (0, l) \subseteq E^2$, where the rectangle is given the trivial S^1-action. This can be interpreted as saying that an open dense subset of the quotient M^4/S^1 admits a Riemannian submersion to $(0, l_1) \times (0, l)$. The geometric idea here is to apply the argument of 2.1.1 and 2.1.2 in the orbit space M^4/S^1. The "totally geodesic" submanifolds will be N^2 and a geodesic segment γ which runs from p_1 to p_2. It turns out that the endpoints of γ don't cause problems in the argument; in fact, they play a crucial role. As in section 3.2, the proof presented below will avoid explicit mention of the orbit space M^4/S^1.

Let $\gamma : [0, l_1] \longrightarrow M^4$ be a minimizing geodesic segment with $\gamma(0) = p_1$, and $\gamma(l_1) = p_2$.

Lemma 3.3.1 *At least one of the two endpoints p_1, p_2 of γ satisfies $dist(p_i, N^2) = dist(Im\gamma, N^2) = dist(S^1(Im\gamma), N^2) = l$.*

Proof: Pick $p \in Im(\gamma)$ closest to N^2, i.e. $dist(p, N^2) = dist(Im\gamma, N^2) = dist(S^1(Im\gamma), N^2)$. If $p = p_1$ or p_2 there is nothing to prove, so assume $p \in$

$interior(Im\gamma)$, and let $\eta_0 : [0,l] \longrightarrow M^4$ be a geodesic from p to $q \in N^2$ with $length(\eta_0) = l = dist(p, N^2)$. The first variation formula shows that η_0 hits $Im(\gamma)$ and N^2 orthogonally, since $length(\eta_0) = dist(p, N^2) = dist(Im\gamma, N^2)$. η_0 is an S^1-geodesic segment since $length(\eta_0) = dist(S^1(p), N^2)$. If $X(0) \in T_pIm\gamma$, then by remark 2.2.6, $X(0)$ can be extended to an S^1-parallel normal vector field $X : [0,l] \longrightarrow TM^4$ along η_0, and it is easy to see that $X(l) \in T_qN^2$ since $X(q) \in (Vert^{S^1}_{\eta_0}(l))^\perp$ (see definition 2.2.5).

Define $\gamma_1 : [0,S] \longrightarrow M^n$ and $\gamma_2 : [0,S] \longrightarrow M^n$ by $\gamma_1(s) = exp(sX(0))$ and $\gamma_2(s) = exp(sX(l))$ respectively, where $S = \sup\{s \in R |\ exp(sX(s)) \in Im(\gamma)\}$. We can apply corollary 2.2.8 to conclude that $dist(\gamma_1(S), N^2) = l$. This establishes the lemma since $\gamma_1(S)$ is an endpoint of $Im(\gamma)$. $\square$

Lemma 3.3.2 *Assume without loss of generality that $dist(p_1, N^2) = dist(Im\gamma, N^2) = l$. Then*

1. *$dist(\cdot, N^2)$ is constant along $Im\gamma$. In particular, $dist(p_2, N^2) = dist(p_1, N^2) = dist(Im\gamma, N^2)$.*

2. *Let $N_1^2 = S^1(Im\gamma)$. Then N_1^2 is a smooth surface.*

3. *Given $p \in N_1^2$ and a geodesic segment $\eta : [0,l] \longrightarrow M^4$ with $\eta'(0) \in \nu_pN_1^2$, we have $\eta(l) \in N^2$. Moreover, if $X(0) \in T_pN_1^2$ satisfies $g(X(0), T_pS^1(p)) = 0$, then $X(0)$ extends to an S^1-parallel, parallel Jacobi vector field along η corresponding to a variation of η through geodesic segments normal to N_1^2.*

Proof: Once again, the idea is to exploit corollary 2.2.8.

Let $\eta_0 : [0,l] \longrightarrow M^4$ be a geodesic segment with $\eta_0(0) = p_1$, $\eta_0(l) \in N^2$, and $length(\eta_0) = dist(p_1, N^2) = dist(Im\gamma, N^2) = l$. By the first variation formula $\sup_{h\in S^1}\{g(h \cdot \eta_0'(0), \gamma'(0))\} \leq 0$, while by lemma 2.3.1 $\sup_{h\in S^1}\{g(h \cdot \eta_0'(0), \gamma'(0))\} \geq 0$. Hence $g(h \cdot \eta_0'(0), \gamma'(0)) = g(\eta_0'(0), h \cdot \gamma'(0)) = 0$ for every $h \in S^1$. Fix $h \in S^1$ for a moment, and extend $X(0) = \gamma'(0)$ to an S^1-parallel vector field $X : [0,l] \longrightarrow TM^4$ along the S^1-minimizing geodesic segment $h \cdot \eta_0$. We have $X(l) \in T_{\eta(l)}N^2$ because $g(X(l), h \cdot \eta_0'(l)) = g(X(l), Vert^{S^1}_{h\cdot\eta_0}(l)) = 0$ and $span((h \cdot \eta_0'(l), Vert^{S^1}_{p\cdot\eta_0}(l)) = \nu_{h\cdot\eta_0(l)}N^2$. Let $S = l_1 = dist(p_1, p_2)$, $\gamma_1 = \gamma : [0,S] \longrightarrow M^4$, $\gamma_2 : [0,S] \longrightarrow M^4$, $\gamma_2(s) = exp(sX(l)) \in N^2$. Since $S^1_{(h\cdot\eta_0)(t)} = S^1_{(h\cdot\eta_0)'(0)} = \{e\}$ for $t \in (0,l)$ by part 3 of lemma 2.2.2, corollary 2.2.8 may be applied to show that $dist(\gamma(s), N^2) = l$ for every $s \in [0, l_1]$, and

to get a totally geodesic isometric immersion $\eta_h : [0, l_1] \times [0, l] \longrightarrow M^4$ which is orthogonal to the S^1 action. For every $s \in [0, l_1]$, $\eta_h \circ (s, id_{[0,l]}) : [0, l] \longrightarrow M^4$ is an S^1– minimizing geodesic segment from $\eta_h(s, 0) = \gamma(s)$ to N^2, and the variation of $\eta_h \circ (s, id_{[0,l]})$ obtained by varying s gives an S^1–parallel, parallel Jacobi vector field along $\eta_h \circ (s, id_{[0,l]})$ with initial condition a multiple of $\gamma'(s)$.

Consider $N_1^2 = S^1(\gamma) \subseteq M^4$. The minimizing segments $h \cdot \gamma$, $h \in S^1$, have initial velocities in the two dimensional space $V = \{(h \cdot \eta'(0)) | \ h \in S^1\}^{\perp} \subseteq T_{p_1} M^4$. Let $D_{l_1} V = \{v \in V | \ |v| < l_1\}$, $\bar{D}_{l_1} V = \{v \in V | \ |v| \leq l_1\}$. Then $N_1^2 = exp(\bar{D}_{l_1} V)$, and $N_1^2 \setminus \{p_2\} = exp(D_{l_1} V)$ =a smooth surface. Analogous reasoning shows that $N_1^2 \setminus \{p_1\}$ is a smooth surface as well. This proves 2.

To prove 3, we may assume that $p = \gamma(s)$ for some $s \in [0, l_1]$. The initial velocities of the minimizing segments $\eta_h \circ (s, id_{[0,l]})$ constructed above fill up $\nu_p N_1^2$ as h runs over S^1 since they lie in a two dimensional subspace of $T_p M^4$ which is orthogonal to $\gamma'(s)$ and $T_p S^1(p)$. Any $X(0) \in T_p N_1^2$ satisfying $g(X(0), T_p S^1(p)) = 0$ must be a multiple of $\gamma'(s)$, so the desired extension $X : [0, l] \longrightarrow TM^4$ of $X(0)$ was constructed above by varying $\eta_h \circ (s, id_{[0,l]})$. □

Lemma 3.3.3 *1. $exp\big|_{D_l \nu N_1^2} : D_l \nu N_1^2 \longrightarrow M^4$ is a diffeomorphism onto $M^4 \setminus N^2$.*

2. S^1 acts freely on $M^4 \setminus (N_1^2 \amalg N^2)$.

Proof: Proceed as in the proof of lemma 3.2.5, except replace the use of lemma 3.2.4 with lemma 3.3.2. □

Corollary 3.3.4 $Fix(S^1, M^4) = \{p_1, p_2\} \amalg N^2$.

Proof: By lemma 3.3.2, $Fix(S^1, M^4) = Fix(S^1, N_1^2 \amalg N^2) = \{p_1, p_2\} \amalg N^2$. □

By lemma 3.3.3 we may define $P = \pi \circ (exp\big|_{D_l \nu N_1^2})^{-1} : M^4 \setminus N^2 \longrightarrow N_1^2$ where $\pi : D_l \nu N_1^2 \longrightarrow N_1^2$ is the bundle projection. Let $\tilde{N}^2 = N_1^2 \setminus \{p_1, p_2\}$ and let $\tilde{\pi} : \tilde{N}^2 \longrightarrow \tilde{N}^2/S^1$ be the quotient map.

Lemma 3.3.5 *$\tilde{P} = \tilde{\pi} \circ (P\big|_{P^{-1}(\tilde{N}^2)} : P^{-1}(\tilde{N}^2) \longrightarrow \tilde{N}^2/S^1$ is an S^1– equivariant Riemannian submersion if $\tilde{N}^2/S^1$ is given the submersion metric induced by $\tilde{\pi} : \tilde{N}^2 \longrightarrow \tilde{N}^2/S^1$.*

Proof: The S^1-equivariance of $\tilde{P}$ follows from that of P and $\tilde{\pi}$. Pick $q \in P^{-1}(\tilde{N}^2)$. There is a unique $p \in \tilde{N}^2$ and a unique geodesic segment $\eta : [0,l] \longrightarrow M^4$ such that $\eta'(0) \in \nu_p\tilde{N}^2$ and $q = \eta(t)$ for some $t \in [0,l)$. Let $X : [0,l] \longrightarrow TM^4$ be a nonzero S^1-parallel, parallel Jacobi vector field along η given part 3 of lemma 3.3.2. Since $|\tilde{P}_*X(t)| = |\tilde{\pi}_*P_*X(t)| = |\tilde{\pi}_*X(0)| = |X(0)| = |X(t)|$, it remains only to show that $X(t)$ is orthogonal to the fiber $\tilde{P}^{-1}(\tilde{P}(q)) = exp(D_l\nu\tilde{N}^2\big|_{S^1(p)})$ along η.

Let $\eta_\lambda : [0,l] \longrightarrow M^4$ be a smooth family of geodesic segments with $\eta'_\lambda(0) \in \nu_p\tilde{N}^2$, $\eta_0 = \eta$. Assume that $Y = \frac{d}{d\lambda}\eta_\lambda\big|_{\lambda=0}$ is a nontrivial Jacobi field along $\eta_0 = \eta$. Then the vertical space of $\tilde{P}$ at q, i.e. $T_q\tilde{P}^{-1}(\tilde{P}(q))$, is spanned by $Y(t)$, $\eta'(t)$, and $T_qS^1(q)$. We already know that $g(X(t),\eta'(t)) = g(X(t),T_qS^1(q)) = 0$, so it suffices to check that $g(X(t),Y(t)) = 0$. Applying the Jacobi equation along η, we have

$$(\frac{d^2}{dt^2}g(X,Y))(t) = g(\nabla^2_{\frac{d}{dt}}Y,X)$$
$$= g(R(\eta'(t),Y(t))\eta'(t),X(t)) = g(R(\eta'(t),X(t))\eta'(t),Y(t))$$
$$= g(\nabla^2_{\frac{d}{dt}}X(t),Y(t)) = 0.$$

Since $g(Y(0),X(0)) = \frac{d}{dt}(g(X(t),Y(t)))(0) = 0$, we have $g(Y(t),X(t)) = 0$ for every $t \in [0,l]$. $\square$

Let $d : M^4 \setminus (N_1^2 \amalg N^2) \longrightarrow (0,l)$ be given by $d(q) = dist(q,N_1^2)$. It follows from lemma 3.3.3 that d is a smooth Riemannian submersion. Let $\tilde{M}^4 = P^{-1}(\tilde{N}^2) \setminus \tilde{N}^2$, and let $\Phi = \tilde{P}\big|_{\tilde{M}^4} \times d\big|_{\tilde{M}^4} : \tilde{M}^4 \longrightarrow \tilde{N}^2/S^1 \times (0,l)$.

Theorem 3.3.6 *Φ is an S^1-equivariant Riemannian submersion.*

Proof: Φ is S^1-equivariant since $\tilde{P}$ and d are S^1-equivariant. To see that Φ is a Riemannian submersion, note that the horizontal distributions of the Riemannian submersions $\tilde{P}\big|_{\tilde{M}^4}$ and $d\big|_{\tilde{M}^4}$ are mutually orthogonal. $\square$

3.4 $Fix(S^1, M^4)$ Contains Four Isolated Points

In this section we study the geometry of a four-dimensional, connected, compact, nonnegatively curved Riemannian manifold (M^4, g) with an isometric S^1 action having at least four isolated fixed points p_1, p_2, p_3, p_4. We will prove

Proposition 3.4.1 *1. The sectional curvature of (M^4, g) is zero for some two-planes.*

2. $Fix(S^1, M^4) = \{p_1, \ldots, p_4\}$.

Part 1 is proved in lemma 3.4.2–3.4.4; the proof of part 2 occupies the remainder of section 3.4.

Lemma 3.4.2 *For each four-tuple $1 \leq i, j, k, l \leq 4$ of distinct integers,*

$$\angle_{p_i}(p_j, p_k) + \angle_{p_i}(p_k, p_l) + \angle_{p_i}(p_l, p_j) \leq \pi.$$

Proof: The notation is explained in definition 2.1.4. Consider $Dir_{p_i}(p_j)$, $Dir_{p_i}(p_k)$, $Dir_{p_i}(p_l) \subseteq S^3T_{p_i}M^4$. These subsets are S^1-invariant, so we may calculate $\angle_{p_i}(Dir_{p_i}(p_j), Dir_{p_i}(p_k))$, etc., by passing to the metric space quotient $(S^3T_{p_i}M^4)/S^1$. The following lemma then refuces the proof of lemma 3.4.2 to checking that if x_1, x_2, $x_3 \in S^2(\frac{1}{2}) \subseteq E^3$ then $\frac{1}{2}(\angle(x_1, x_2) + \angle(x_2, x_3) + \angle(x_3, x_1)) \leq \pi$. $\square$

Lemma 3.4.3 *Let $S^1 \longrightarrow O(4)$ be an orthogonal linear representation with no trivial direct summands, and let $(S^3(1)/S^1, \angle)$ denote the quotient space under this S^1 action. Then $(S^3(1)/S^1, \angle)$ admits a distance nondecreasing map to $(S^2(\frac{1}{2}), \frac{1}{2}\angle)$, where $S^2(\frac{1}{2}) = \{x \in E^3 | \, |x| = \frac{1}{2}\}$, and $\frac{1}{2}\angle$ is the distance fuction of the Riemannian metric on $S^2(\frac{1}{2})$ induced by the inclusion $S^2(\frac{1}{2}) \subseteq E^3$.*

Proof: An orthogonal representation $S^1 \times E^4 \longrightarrow E^4$ of S^1 is orthogonally equivalent to a representation $\Psi_{k,l} : S^1 \times C^2 \longrightarrow C^2$:

$$\alpha \cdot (z_1, z_2) = (\alpha^k z_1, \alpha^l z_2)$$

where $\alpha \in S^1 \subseteq C$, $(z_1, z_2) \in C^2$, and $k, l \in Z$ are appropriately chosen. Such a representation has no fixed subspaces iff $k, l \neq 0$. Let $(\Sigma_{k,l}, \angle_{k,l})$ denote the quotient of S^3 by the action $\Psi_{k,l}$.

It is easy to check that $(\Sigma_{1,1}, \angle_{1,1})$ is isometric to $(S^2(\frac{1}{2}), \frac{1}{2}\angle)$, where $\frac{1}{2}\angle$ is the Riemannian distance on $S^2(\frac{1}{2})$; the isometry is induced by the Hopf map $S^3(1) \longrightarrow S^2(\frac{1}{2})$.

Let $\widetilde{S^3(1)} = \{(z_1, z_2) \in C^2 | \ z_1 z_2 \neq 0\}$. $\widetilde{S^3(1)}$ is $\Psi_{k,l}$ invariant, and the restriction of the action $\Psi_{k,l}$ to $\widetilde{S^3(1)}$ has only one orbit type, so $\tilde{\Sigma}_{k,l} \stackrel{def}{=} \widetilde{S^3(1)}/S^1$ inherits a canonical smooth submersion Riemannian metric $h_{k,l}$ from the Riemannian manifold $\widetilde{S^3(1)} \subseteq E^4$. The distance function induced by this Riemannian metric coincides with $\angle_{k,l}\big|_{\tilde{\Sigma}_{k,l}}$. Note that $(\tilde{\Sigma}_{k,l}, h_{k,l})$ is a surface of revolution since the $T^2 = S^1 \times S^1$ action $T^2 \times C^2 \longrightarrow C^2$ given by $(\alpha, \beta) \cdot (z_1, z_2) = (\alpha z_1, \beta z_2)$ induces an isometric T^2/S^1 action on $(\tilde{\Sigma}_{k,l}, h_{k,l})$. The meridians on $(\tilde{\Sigma}_{k,l}, h_{k,l})$ correspond to (i.e. have as inverse image) T^2 orbits in $\widetilde{S^3(1)}$. If a given meridian has inverse image $T^2((z_1, z_2))$, $(z_1, z_2) \in \widetilde{S^3(1)}$, then its length is

$$\frac{2\pi |z_1||z_2|}{\sqrt{k^2|z_1|^2 + l^2|z_2|^2}}$$

Construct a map $\tilde{\phi}_{k,l}$ from the surface of revolution $\tilde{\Sigma}_{k,l}$ to the surface of revolution $\tilde{\Sigma}_{1,1}$ which carries longitudes to longitudes, commutes with "revolution," and preserves the correspondence of meridians with T^2 orbits in $\widetilde{S^3(1)}$. The calculation above shows that $(\tilde{\phi}_{k,l})^* h_{1,1} \geq h_{k,l}$, so

$$\tilde{\phi}_{k,l} : (\tilde{\Sigma}_{k,l}, \angle_{k,l}) \longrightarrow (\tilde{\Sigma}_{1,1}, \angle_{1,1})$$

is a distance nondecreasing map. Extending the Lipshitz map $\tilde{\phi}_{k,l}$ continuously to $\phi_{k,l} : \Sigma_{k,l} \longrightarrow \Sigma_{1,1}$, we get a distance nondecreasing map

$$(S^3(1)/S^1, \angle) \approx (\Sigma_{k,l}, \angle_{k,l}) \longrightarrow (\Sigma_{1,1}, \angle_{1,1}) \approx (S^2(\frac{1}{2}), \frac{1}{2}\angle).$$

□

Lemma 3.4.4 *1. For each triple $1 \leq i, j, k \leq 4$ of distinct integers $\angle_{p_i}(p_j, p_k) + \angle_{p_j}(p_k, p_i) + \angle_{p_k}(p_i, p_j) = \pi$.*

2. (M^4, g) doesn't have strictly positive curvature.

3. For each quadruple of distinct integers $1 \leq i, j, k, l \leq 4$, $\angle_{p_i}(p_j, p_k) + \angle_{p_i}(p_k, p_l) + \angle_{p_i}(p_l, p_j) = \pi$.

Proof: View the four points $p_1, \ldots p_4$ as the vertices of a tetrahedron, and add up all the angles. We have

$$\sum_{1\le i\le 4}\sum_{j\ne i}\sum_{k\ne i,k>j} \angle_{p_i}(p_j,p_k)$$

$$\le 4\pi$$

by lemma 3.4.2, while by theorem 2.1.5

$$\sum_{1\le i<j<k\le 4} (\angle_{p_i}(p_j,p_k)+\angle_{p_k}(p_i,p_j)+\angle_{p_j}(p_k,p_i)) \ge 4\pi$$

Hence equality must hold everywhere, which implies the lemma. □

To prove part 2 of proposition 3.4.1 indirectly, assume henceforth that $Fix(S^1, M^4)$ contains $\{p_1, \ldots, p_4\}$ strictly. From the discussion in section 3.1, $Fix(S^1, M^4)$ is a disjoint union of totally geodesic surfaces and isolated points, and section 3.3 rules out the possibility that $Fix(S^1, M^4)$ contains surfaces components. Therefore $Fix(S^1, M^4)$ must contain an additional isolated point, say p_5.

Lemma 3.4.5 *Fix $1 \le i \le 5$.*

1. *The sets $Dir_{p_i}(p_j) \subseteq S^3T_{p_i}M^4$, $j \ne i$ consist of a single S^1 orbit each, and come in mutually orthogonal pairs.*

2. *The isotropy representation of S^1 on $T_{p_i}M^4$ is orthogonally equivalent to the action of $S^1 \subseteq C$ on C^2 by scalar multiplication.*

Proof: For notational convenience, take $i = 5$. The reasoning of lemmas 3.4.2-3.4.4 applied to the four isolated fixed points p_5, p_j, p_k, p_l implies that

$$\angle_{p_5}(p_j,p_k)+\angle_{p_5}(p_k,p_l)+\angle_{p_5}(p_l,p_j)=\pi$$

for each triple of distinct integers $1 \le j, k, l \le 4$. Equivalently,

$$\angle_{p_5}(Dir_{p_5}(p_j), Dir_{p_5}(p_k))+\angle_{p_5}(Dir_{p_5}(p_k), Dir_{p_5}(p_l))+\angle_{p_5}(Dir_{p_5}(p_l), Dir_{p_5}(p_j))$$

$$=\pi.$$

Let $(S^3T_{p_3}M^4)/S^1, \bar{\angle}_{p_3})$ be the metric space quotient of $(S^3T_{p_3}M^4, \angle_{p_3})$ and let $\phi : (S^3T_{p_3}M^4)/S^1 \longrightarrow S^2(\frac{1}{2})$ be the distance nondecreasing map provided by lemma 3.4.3. Let $D_m = \phi \circ \pi(Dir_{p_3}(p_m)) \subseteq S^2(\frac{1}{2})$, $m = 1, \ldots, 4$, where $\pi : S^3T_{p_3}M^4 \longrightarrow S^3T_{p_3}M^4/S^1$ is the quotient map. For each triple $1 \leq i, j, k \leq 4$ of distinct integers,

$$\frac{1}{2}\angle(D_i, D_j) + \frac{1}{2}\angle(D_j, D_k) + \frac{1}{2}\angle(D_k, D_i) \geq \pi.$$

This clearly forces the sets D_m to lie on a great circle, to consist of a single point each, and to pair off as antipodal points, i.e. $\cup_{m=1}^4 D_m$ consists of two pairs of antipodal points. This proves part 1 of the lemma.

Part 2 of lemma 3.4.4 follows easily from lemma 2.3.1 and the fact that the isotropy representation of S^1 on $T_{p_3}M^4$ contains more than one pair of mutually orthogonal orbits in $S^3T_{p_3}M^4$. $\square$

We now show that if $\angle_{p_1}(p_j, p_k) = \frac{\pi}{2}$, then $Dir_{p_j}(p_k)$ cannot be a single S^1 orbit in $S^3T_{p_j}M^4$, in contradiction to lemma 3.4.5. Assume—after relabelling the points if necessary—that

$$\angle_{p_1}(p_2, p_3) = \angle_{p_1}(Dir_{p_1}(p_2), Dir_{p_1}(p_3)) = \frac{\pi}{2}.$$

If $\gamma_2 \in Min_{p_1}(p_2)$, $\gamma_3 \in Min_{p_1}(p_3)$, (see definition 2.1.4) then $\angle_{p_1}(\gamma_2'(0), \gamma_3'(0)) = \frac{\pi}{2}$ by lemma 3.4.5. By part 2 of theorem 2.1.5, there is a flat, totally geodesic, triangular surface $\Delta^2 \subseteq M^4$ with geodesic edges γ_2, γ_3, and $\eta \in Min_{p_2}(p_3)$. If we replace γ_3 with $h \cdot \gamma_3$ for some $h \in S^1$, then Toponogov's theorem will produce another flat, totally geodesic, triangular surface $\Delta_h^2 \subseteq M^4$ with geodesic edges γ_2, $h \cdot \gamma_3$, and $\eta_h \in Min_{p_2}(p_3)$. In particular, $\angle_{p_2}(\gamma_2, \eta_h) = \angle_{p_2}(\gamma_2, \eta) = \angle_{p_2}(p_1, p_3) < \frac{\pi}{2}$.

Let $v = -\gamma_2'(l_{12}) \in Dir_{p_2}(p_1) \subseteq S^3T_{p_2}M^4$, where $l_{12} = dist(p_1, p_2)$, and let w, $w_h \in Dir_{p_2}(p_3) \subseteq S^3T_{p_2}M^4$ be the initial velocities of η and η_h, respectively. We have $\angle_{p_2}(v, w) = \angle_{p_2}(v, w_h) = \angle_{p_2}(v, Dir_{p_2}(p_3)) < \frac{\pi}{2}$. The following lemma forces $w = w_h$ since by lemma 3.4.5 $Dir_{p_2}(p_3) \subseteq S^3T_{p_2}M^4$ consists of a single S^1 orbit and the isotropy representation at p_2 is equivalent to the standard representation $\Psi_{1,1}$ (see lemma 3.4.3). Since $\eta'(0) = w = w_h = \eta_h'(0)$ we have $\eta_h = \eta$, which forces $\Delta_h^2 = \Delta^2$. This is impossible unless $h = e \in S^1$, so we have a contradiction.

Lemma 3.4.6 *Let $S^1 \subseteq C$ act on C^2 by scalar multiplication, and suppose $v, w \in S^3(1) \subseteq C^2$. Then either $\angle(S^1(v), S^1(w)) = \frac{\pi}{2}$ or there exists a unique $h \in S^1$ such that $\angle(v, h \cdot w) = \angle(S^1(v), S^1(w)) = \angle(v, S^1(w))$.*

Proof: The Riemannian submersion metric on $S^3(1)/S^1$ is isometric to $S^2(\frac{1}{2}) \subseteq E^3$. The orbits $S^1(v)$, $S^1(w) \in S^3(1)/S^1$ are either separated by $\frac{\pi}{2} = diameter(S^2(\frac{1}{2}))$, or they are joined by a unique minimizing geodesic segment $\bar{\gamma} \subseteq S^3(1)/S^1$ with $length(\bar{\gamma}) = \angle(S^1(v), S^1(w))$. If $w_0 \in S^1(w)$ satisfies $\angle(v, w_0) = \angle(S^1(v), S^1(w))$, and γ is a minimizing geodesic segment from v to w_0, then γ projects to the minimizing curve $\bar{\gamma}$. Hence γ is the unique horizintal lift of $\bar{\gamma}$ starting at v, and w_0 is unique. $\square$

4 Nonnegatively Curved Riemannian Four-manifolds with $SU(2)$ Symmetry

In this section, we study the geometry of compact, one-connected, nonnegatively curved, Riemannian four-manifolds (M^4, g) which admit almost effective isometric $SU(2)$ actions. We start by determining the underlying smooth $SU(2)$ manifolds.

Notation 4.0.7 *$\overline{SO(2)}$, $\overline{O(2)} \subseteq SU(2)$ will denote $\pi^{-1}(SO(2))$ and $\pi^{-1}(O(2))$ respectively, where $\pi : SU(2) \longrightarrow SO(3)$ is the usual two-fold covering homomorphism and $SO(2)$, $O(2) \subseteq SO(3)$ are the usual inclusions.*

Smooth Classification 4.0.8 *Let (M^4, ρ) be a smooth, almost effective (i.e. $ker\rho$ is discrete) $SU(2)$ action $\rho : SU(2) \times M^4 \longrightarrow M^4$ on a compact one-connected four-manifold M^4. Then (M^4, ρ) is smoothly equivalent to one of the $SU(2)$ manifolds below:*

I. $(S^4, (\tilde{\rho}_3 \oplus 2\theta)\big|_{S^4})$. Here $\tilde{\rho}_3$ denotes the pullback of the standard three-dimensional irreducible representation ρ_3 of $SO(3)$ to $SU(2)$, and θ is a one-dimensional trivial representation. The orbit space $S^4/SU(2)$ is homeomorphic to D^2, and the orbit diagram is:

II. $(S^2 \times S^2, (3\theta \oplus \tilde{\rho}_3)\big|_{S^2 \times S^2})$. *$3\theta \oplus \tilde{\rho}_3$ is a representation on R^6 which preserves the splitting $R^6 \approx R^3 \oplus R^3$, so it can be restricted to $S^2 \times S^2 \subseteq R^3 \times R^3$. $S^2 \times S^2/SU(2) \approx S^2$, and all the orbits are principal orbits with principal isotropy group conjugate to $\widetilde{SO(2)}$.*

III. $(S^4, (\mu_2 \oplus \theta)\big|_{S^4})$. *$\mu_2$ is the four-dimensional irreducible representation of $SU(2)$. $S^4/SU(2) \approx I = [0,1]$. The orbit diagram is:*

IV. $(CP^2, \overline{\mu_2 \oplus \theta_C})$. *View μ_2 as a two-dimensional complex representation and let θ_C be a one dimensional trivial complex representation. Let $\overline{\mu_2 \oplus \theta_C}$ be the action induced on CP^2 by the complex representation $\mu_2 \oplus \theta_C$. The orbit diagram is:*

V. $(S^4, (s^2\tilde{\rho}_3 - 1)\big|_{S^4})$. *$s^2\tilde{\rho}_3$ is the symmetric square of $\tilde{\rho}_3$, and $s^2\tilde{\rho}_3 - 1$ is the five-dimensional irreducible summand of $s^2\tilde{\rho}_3$. The orbit diagram is:*

VI. $(CP^2, \overline{(\tilde{\rho}_3)_C})$. *$(\tilde{\rho}_3)_C$ is the complexification of $\tilde{\rho}_3$, and it induces the action $\overline{(\tilde{\rho}_3)_C}$ on CP^2. The orbit diagram:*

VII. $(SU(2) \times_{\phi_n} S^2, \overline{left})$. *Let $\phi_n : S^1 \times S^2 \longrightarrow S^2$ be an S^1 action with ineffective kernel $Z_n \subseteq S^1$, $n \geq 1$, $Z_1 = \{e\}$. Construct the asociated S^2 bundle $SU(2) \times_{\phi_n} S^2$ to the S^1 principal bundle $S^1 \longrightarrow SU(2) \longrightarrow SU(2)/S^1$. Left translation on $SU(2)$ induces an $SU(2)$ action $\overline{left}$ on the associated bundle $SU(2) \times_{\phi_n} S^2$. The orbit diagram is:*

Note that $SU(2) \times_{\phi_n} S^2 \approx S^2 \times S^2$ when n is even, and $SU(2) \times_{\phi_n} S^2 \approx CP^2 \# -CP^2$ when n is odd. To see this, observe that an S^2 bundle over $SU(2)/S^1 \approx S^2$ is determined by a transition function in $\pi_1(SO(3)) \approx Z_2$.

Sketch of the proof of the classification: Fix an almost effective, compact, one-connected $SU(2)$ manifold (M^4, ρ). Let $G = SU(2)$, and let H be the principal isotropy subgroup for the action ρ. H is well defined up to conjugacy in G. As before, $G(p)$ and G_p denote the G orbit and the isotropy subgroup of $p \in M^4$, respectively.

$SU(2)$ contains no two-dimensional subgroups, so $dim(H) = 0$ or $dim(H) = 1$. We analyze these two cases separately.

Case A: $Dim(H) = 1$. This gives $H \approx \widetilde{SO(2)}$ or $H \approx \widetilde{O(2)}$. We consider $H \approx \widetilde{SO(2)}$ first. Nonprincipal isotropy subgroups can only be conjugates of $\widetilde{O(2)} \subseteq SU(2)$ or $SU(2)$ itself.

If $G_p \approx \widetilde{O(2)}$ then the slice representation of G_p on $\nu_p G(p)$ has as principal isotropy subgroup a conjugate of $H \approx \widetilde{SO(2)}$. This forces the slice representation to factor through a representation of $\widetilde{O(2)}/\widetilde{SO(2)} \approx Z_2$ on $\nu_p G(p)$. There are only two possibilities for such a Z_2 representation: reflection or -1. The corresponding local structures for the orbit space are $R^2/reflection \approx R \times R_+$ =a half space with $\widetilde{O(2)}$-type orbits on the boundary, and $R^2/inversion \approx R^2$ with an isolated $\widetilde{O(2)}$-type orbit.

If $G_p \approx SU(2)$ then the isotropy representation of $SU(2)$ on T_pM^4 is equivalent to $\tilde{\rho}_3 \oplus \theta$ since its principal isotropy subgroup is conjugate to $H \approx \widetilde{SO(2)}$. Near $G(p)$, $M^4/SU(2)$ is homeomorphic to $R^4/\tilde{\rho}_3 \oplus \theta \approx R^3/\tilde{\rho}_3 \times R^1 \approx R_+ \times R$, so M^4/G is locally a manifold with boundary, the orbits near $G(p)$ being either fixed points or principal orbits.

M^4/G is a surface with boundary. Boundary components are made up of either fixed points, or $\widetilde{O(2)}$-type orbits. We can rule out the latter possibility since the union of the $\widetilde{O(2)}$-type orbits along a boundary component of M^4/G would be a one-sided hypersurface in M^4, which is impossible since $\pi_1(M^4) = \{e\}$.

If M^4 contains two isolated $\widetilde{O(2)}$-type orbits, then one can find a path in the surface M^4/G joining them whose inverse image in M^4 is a one-sided hypersurface. So M^4 contains at most one $\widetilde{O(2)}$-type orbit. But M^4 contains an even number of isolated $\widetilde{O(2)}$-type orbits. To see this, notice that isolated $\widetilde{O(2)}$-type orbits are in one-to-one correspondence with isolated points in $Fix(\widetilde{O(2)}, M^4)$. Since $\widetilde{SO(2)} \lhd \widetilde{O(2)}$, $\widetilde{O(2)}$ preserves $Fix(\widetilde{SO(2)}, M^4)$, and

we have $Fix(\widetilde{O(2)}, M^4) = Fix(\widetilde{O(2)}, Fix(\widetilde{SO(2)}, M^4))$. Smith theory gives $\chi(Fix(\widetilde{O(2)}, Fix(\widetilde{SO(2)}, M^4))) \equiv \chi(Fix(\widetilde{SO(2)}, M^4)) mod\ 2 \equiv 0\ mod\ 2$ because $Fix(\widetilde{SO(2)}, M^4)$ is an orientable surface. But $Fix(\widetilde{O(2)}, M^4)$ is a disjoint union of circles and isolated points; therefore the number of isolated $\widetilde{O(2)}$-type orbits is even.

To summarize, the only nonprincipal isotropy subgroup possible is $SU(2)$. The only possibilities for the orbit diagram are:

because $\pi_1(M^4/G$ is trivial. One can easily check that these orbit diagrams determine the $SU(2)$ manifolds up to smooth $SU(2)$ equivalence.

Now suppose $H \approx \widetilde{O(2)}$. The only possibility for a nonprincipal isotropy subgroup is $SU(2)$, and this can be ruled out by observing that the isotropy representation couldn't have the correct principal isotropy subgroup. Hence the orbits are all principal. But this forces $M^4 \longrightarrow M^4/SU(2)$ to be a trivial fiber bundle with fiber RP^2, which contradicts $\pi_1(M^4) = \{e\}$. Hence H cannot be conjugate to $\widetilde{O(2)}$.

Case B: $Dim(H) = 0$. This gives $dim(M^4/SU(2)) = 1$, and $M^4/SU(2)$ is homeomorphic to S^1 or to a closed interval, depending on whether all orbits are principal or not. Since $\pi_1(M^4/SU(2))$ is trivial, the former case is impossible because it would give a fiber bundle $M^4 \longrightarrow S^1$. Therefore there are two nonprincipal orbits, corresponding to the endpoints of the interval $M^4/SU(2)$. Neither of these can be an exceptional orbit, because that would produce a one-sided hypersurface in M^4. So the nonprincipal orbits $G(p_1)$, $G(p_2)$ are both singular, with G_{p_i} conjugate to $\widetilde{SO(2)}$, $\widetilde{O(2)}$, or $SU(2)$.

If G_{p_i} is conjugate to $\widetilde{SO(2)}$, then $dim(\nu_{p_i}G(p_i) = 2)$ and the slice representation of G_{p_i} is fixed by its principal isotropy subgroup $\approx Z_n$.

If $G_{p_i} \approx \widetilde{O(2)}$, then $dim(\nu_{p_i}G(p_i) = 2)$, and one can check that the slice representation of G_{p_i} on $\nu_{p_i}G(p_i)$ must factor through the two-fold covering homomorphism $\pi\big|_{\widetilde{O(2)}} : \widetilde{O(2)} \longrightarrow O(2)$. The two-dimensional representations of $O(2)$ with discrete kernel are indexed by their dihedral principal isotropy groups $D_n \subseteq O(2)$. We denote the corresponding principal isotropy subgroups in $\widetilde{O(2)}$ by $\tilde{D}_n$.

If $G_{p_1} = SU(2)$ then the isotropy representation must be $\approx \mu_2$ since $dim(H) = 0$. Hence $H = \{e\}$.

Now we consider the possibilities for the triple (H, G_{p_1}, G_{p_2}). If $G_{p_1} = SU(2)$ then $H = \{e\}$, which forces $G_{p_2} = SU(2)$ or $G_{p_2} = \widetilde{SO(2)}$ with G_{p_2} acting effectively on the slice $\nu_{p_2}G(p_2)$. Both of these possibilities occur as actions III and IV.

If G_{p_1} is conjugate to $\widetilde{O(2)}$, then H is conjugate to $\tilde{D}_n$, so either G_{p_2} is conjugate to $\widetilde{O(2)}$ or G_{p_2} is conjugate to $SO(2)$ and $n = 1$. These possibilities are realized in actions V and VI. When G_{p_2} is conjugate to $\widetilde{O(2)}$, the only way to achieve $\pi_1(M^4) = \{e\}$ is to have H conjugate to $\tilde{D}_2$.

All of the actions above are determined up to smooth $SU(2)$ equivalence by the triple (H, G_{p_1}, G_{p_2}), together with the condition that $\pi_1(M^4) = \{e\}$. $\square$

All of the $SU(2)$ manifolds in the classification above admit obvious positively curved invariant metrics except those of type II and VII. The results of section 3 imply that $SU(2)$ manifolds of type II and VII do not admit such metrics, but we give a direct proof here. We will show that an invariant metric on an $SU(2)$ manifold of type II or VII carries disjoint totally geodesic surfaces. Theorem 2.1.1 then implies that such metrics don't have positive curvature.

Consider first an invariant metric on an $SU(2)$ manifold of type II. $Fix(\widetilde{SO(2)}, S^2 \times S^2)$ is a disjoint union of two surfaces, and it must be totally geodesic in $S^2 \times S^2$ relative to any $SU(2)$-invariant metric.

Now consider an invariant metric on an $SU(2)$ manifold of type VII. The two singular orbits are totally geodesic provided $n \neq 4$. If $G(p)$ is one of the singular orbits, then G_p is conjugate to $\widetilde{SO(2)}$. Denote the representation of G_p on R^2 with ineffective kernel Z_n by $R^2_{(n)}$. The isotropy representation of G_p on $T_pG(p)$ is equivalent to $R^2_{(2)}$, while the slice representation of G_p on $\nu_pG(p)$ is equivalent to $R^2_{(n)}$. The second fundamental form of $G(p)$ at p, II_p, is invariant under the action of G_p, i.e.

$$\begin{aligned} II_p &\in Hom_{G_p}(s^2(T_pG(p)), \nu_pG(p)) \\ &\approx Hom_{G_p}(s^2(R^2_{(2)}), R^2_{(n)}) \\ &\approx Hom_{G_p}(R^1 \oplus R^2_{(4)}, R^2_{(n)}) \end{aligned}$$

$$\approx Hom_{G_p}(R^1, R^2_{(n)}) \oplus Hom_{G_p}(R^2_{(4)}, R^2_{(n)})$$
$$\approx \{0\}$$

unless $n = 4$.

On the other hand, if $n > 2$ then $Fix(H, M^4)$ is a surface with two components. Again, this is totally geodesic with respect to any invariant metric.

Theorem 4.0.9 *Let g be an $SU(2)$-invariant metric on an $SU(2)$-manifold of type II. If g has nonnegative curvature, then the $SU(2)$-Riemannian manifold $(S^2 \times S^2, (3\theta \oplus \tilde{\rho}_3)\big|_{S^2 \times S^2}, g)$ is $SU(2)$- equivalent to a Riemannian product of (S^2, g_1) and (S^2, g_2), where g_1 has nonnegative curvature, g_2 has constant curvature, and $SU(2)$ acts on the second factor of the Riemannian product $S^2, g_1) \times (S^2, g_2)$.*

Proof: We will view this action as an effective $SO(3)$ action on $S^2 \times S^2$. Let g be an $SO(3)$-invariant metric on $S^2 \times S^2$ with the action $h \cdot (x, y) = (x, h \cdot y)$, $(x, y) \in S^2 \times S^2$, $h \in SO(3)$. To show that g is a product metric, we show that the horizontal subspaces $\pi_2^{-1}(y)$, $y \in S^2$ intersect the vertical subspaces $\pi_1^{-1}(x)$, $x \in S^2$, orthogonally, and we find metrics g_1 and g_2 on S^2 so that $\pi_i : (S^2 \times S^2, g) \longrightarrow (S^2, g_i)$, $i = 1, 2$ are Riemannian submersions.

If $p = (x, y) \in S^2 \times S^2$, then $(SO(3))_p \approx SO(2)$ preserves the submanifolds $S^2 \times \{y\} = \pi_2^{-1}(y)$ and $\pi_1^{-1}(x)$, so the isotropy representation of $(SO(3))_p$ preserves $T_p\pi_2^{-1}(y)$ and $T_p\pi_1^{-1}(x)$. Having no $(SO(3))_p$ irreducible summands in common, these two subspaces are $g(p)$ orthogonal, i.e. $\pi_2^{-1}(y)$ intersects $\pi_1^{-1}(x)$ orthogonally.

Identify $\pi_1 : S^2 \times S^2 \longrightarrow S^2$ with the quotient map $S^2 \times S^2 \longrightarrow (S^2 \times S^2)/SO(3) \approx S^2 \times (S^2/SO(3)) \approx S^2$ and let g_1 be the submersion metric on S^2 induced by π_1. Then clearly $\pi_1 : (S^2 \times S^2 2, g) \longrightarrow (S^2, g_1)$ is a Riemannian submersion.

It remains only to show that there is a metric g_2 on S^2 such that $\pi_2 : (S^2 \times S^2, g) \longrightarrow (S^2, g_2)$ is a Riemannian submersion. To prove this it is sufficient to show that the fibers $\pi_2^{-1}(y_1)$ and $\pi_2^{-1}(y_2)$ are equidistant for nearby y_1 and y_2, i.e. $dist(\cdot, \pi_2^{-1}(y_2))$ and $dist(\pi_2^{-1}(y_1), \cdot)$ are constant functions on $\pi_2^{-1}(y_1)$ and $\pi_2^{-1}(y_2)$ respectively.

Note: g is invariant under the $O(3)$ action $h \cdot (x,y) = (x, h \cdot y)$, $(x,y) \in S^2 \times S^2$, $h \in O(3)$.

Proof of Note: The $O(3)$ orbits are the same as the $SO(3)$ orbits, so it suffices to check that $(O(3))_p \approx O(2)$ preserves $g(p) \in s^2(T_p^*(S^2 \times S^2))$ for every $p \in S^2 \times S^2$. But this follows easily from the fact that $(SO(3))_p \approx SO(2)$ preserves $g(p)$.

Given y_1, $y_2 \in S^2$ there is a reflection $r \in O(3)$ which fixes y_1 and y_2. Therefore $Fix(r, S^2 \times S^2) = S^2 \times Fix(r, S^2) \approx S^2 \times S^1$ is a totally geodesic, nonnegatively curved submanifold containing $\pi_2^{-1}(y_1)$ and $\pi_2^{-1}(y_2)$. Lemma 2.1.3 shows that the totally geodesic surfaces $\pi_2^{-1}(y_1)$ and $\pi_2^{-1}(y_2)$ must be equidistant inside $Fix(r, S^2 \times S^2)$. It follows that $\pi_2^{-1}(y_1)$ and $\pi_2^{-1}(y_2)$ are equidistant in $S^2 \times S^2$ if y_1 and y_2 are close enough. $\square$

We now investigate the structure of nonnegatively curved invariant metrics on $SU(2)$ manifolds of type VII. Recall that n is the order of the principal isotropy subgroup of this action.

Let $G(q_1)$ and $G(q_2)$ be the two singular orbits and let $l = dist(G(q_1), G(q_2))$. Assume the $q_i \in G(q_i)$ have been chosen so that $dist(q_1, q_2) = l$.

Lemma 4.0.10 *Let $\gamma : [0,l] \longrightarrow M^4$ be an $SU(2)$-geodesic segment with $\gamma(0) = q_1$, $\gamma(l) = q_2$. If $n > 2$ then parallel transport along $Im\gamma$ maps $T_{q_1}G(q_1)$ to $T_{q_2}G(q_2)$.*

Proof: Let $G_\gamma = \{h \in G |\ h \ fixes\ Im(\gamma)\}$. The uniqueness of parallel transport along $Im\gamma$ implies that parallel transport along γ gives an orthogonal transformation $P_{0l} : T_{\gamma(0)}M^4 \longrightarrow T_{\gamma(l)}M^4$ which is equivariant with respect to the isotropy representations of G_γ on $T_{q_1}M^4$ and $T_{q_2}M^4$. For all n, $G_\gamma = \{h \in G |\ h_*\gamma'(0) = \gamma'(0)\}$ = a conjugate of H, while for $n > 2$ the isotropy representations of G_γ on $T_{q_1}M^4$ and $T_{q_2}M^4$ have fixed subspaces $\nu_{q_1}G(q_1)$ and $\nu_{q_2}G(q_2)$ respectively. This forces $P_{0l}(T_{q_1}G(q_1)) = T_{q_2}G(q_2)$.

Alternatively, note that if $n > 2$, then $N^2 = Fix(G_\gamma, M^4)$ is a disjoint union of two totally geodesic S^2's, one of which contains $Im\gamma$. Hence parallel transport along $Im\gamma$ preserves TN^2 and νN^2. Since N^2 intersects $G(q_1)$ and $G(q_2)$ orthogonally, $P_{0l}(T_{q_1}G(q_1)) = T_{q_2}G(q_2)$. $\square$

Lemma 4.0.11 *Let q_1, q_2, and γ be as in lemma 4.0.10. Then*

1. *Any parallel field $X : [0,l] \longrightarrow \nu(Im\gamma)$ with $X(0) \in T_{q_1}G(q_1)$ is the restriction to $Im\gamma$ of an $SU(2)$- Killing vector field.*

2. *Any such X is orthogonal to $exp(D_l\nu_{q_1}G(q_1)) = G_{q_1}(Im\gamma) = \cup_{h\in G_{q_1}} h \cdot Im\gamma$.*

Proof: Consider the family of equidistant hypersurfaces $N_t = \{x \in M^4 | \, dist(x, G(q_1)) = t\}$, $t \in (0,l)$. $Im\gamma$ is normal to this family, so we can apply the Riccati equation to the second fundamental forms of the N_t along $Im\gamma$. This implies, for $t \in (0,l)$, $X : [0,l] \longrightarrow \nu(Im\gamma)$, X parallel along $Im\gamma$, that $\frac{d}{dt}(g(II_{N_t}(X(t),X(t)),\gamma'(t))) \geq 0$ with equality only if $II_{N_t}(X(t),Y) = 0$ for all $Y \in \nu_{\gamma(t)}Im\gamma$.

Riccati for general.

If $X(0) \in T_{q_1}G(q_1)$, then $X(l) \in T_{q_2}G(q_2)$, so

$$\begin{aligned} & g(II_{G(q_1)}(X(0),X(0)),\gamma'(0)) \\ &= \lim_{t\to 0} g(II_{N_t}(X(t),X(t)),\gamma'(t)) \\ &\leq \lim_{t\to l} g(II_{N_t}(X(t),X(t)),\gamma'(t)) \\ &= g(II_{G(q_2)}(X(l),X(l)),\gamma'(l)). \end{aligned}$$

If we take $X_1(0)$, $X_2(0) \in T_{q_1}G(q_1)$ orthonormal, then $X_1(l)$, $X_2(l) \in T_{q_2}G(q_2)$ will be orthonormal and $0 = g(H_{G(q_1)},\gamma'(0))$

$$\begin{aligned} &= g(\sum_{i=1,2} II_{G(q_1)}(X_i(0),X_i(0)),\gamma'(0)) \\ &\leq g(\sum_{i=1,2} II_{G(q_2)}(X_i(l),X_i(l)),\gamma'(l)) \\ &= g(H_{G(q_2)},\gamma'(l)) = 0 \end{aligned}$$

where $H_{G(q_i)}$ denotes the mean curvature of the orbit $G(q_i)$, and the singular orbits $G(q_1)$ and $G(q_2)$ are minimal. This forces

$$\diamond \qquad II_{N_t}(X_i(t),Y) = 0$$

for all $t \in (0,l)$, $i = 1,2$, $Y \in \nu_{\gamma(t)}Im\gamma$.

A vector field $X : (0,l) \longrightarrow \nu(Im\gamma)$ along $Im\gamma$ is a Jacobi field corresponding to a variation through geodesics normal to the N_t if and only if

$$\nabla_{\gamma'(t)}X = W(t)X(t)$$

where $W(t) : T_{\gamma(t)}N_t \longrightarrow T_{\gamma(t)}N_t$ is the Weingarten map of N_t, or equivalently, if

$$g(\nabla_{\gamma'(t)}X, Y) = -g(II_{N_t}(X(t), Y), \gamma'(t))$$

for every $t \in (0, l)$, $Y \in \nu_{\gamma(t)}Im\gamma$. From $\diamond$ it is clear that the X_i, $i = 1, 2$, are Jacobi fields along $Im\gamma$ corresponding to variations through geodesics normal to the X_t, and since $SU(2)$ acts transitively on N_t for each t, the X_i are induced by Killing fields.

We now show that the X_i are normal to the disk $exp(D_l\nu_{q_1}G(q_1)) = G_{q_1}(\gamma([0, l])$. Pick $t_0 \in (0, l)$, and solve for $Y : (0, l) \longrightarrow \nu(Im\gamma)$ with $g(Y(t_0), X_i(t)) = 0$, $|Y(t_0)| \neq 0$, and $(\nabla_{\gamma'(t)}Y)(t) = W(t)Y(t)$. Then as before, Y is the restriction to $Im\gamma$ of an $SU(2)$ Killing field on M^4. What's more, $g(Y(t), X_i(t)) = 0$ for every $t \in (0, l)$ since $\frac{d}{dt}(g(Y(t), X(t)))$

$$= g(\nabla_{\gamma'(t)}Y, X_i(t)) = -g(II_{N_t}(Y(t), X_i(t)), \gamma'(t)) = 0.$$

Hence the Killing field which restricts to Y is zero at q_1—for otherwise the Killing fields which restrict to Y, X_1, X_2 would span a three-dimensional subspace of $T_{q_1}M^4$—and must therefore arise from the action of the isotropy group G_{q_1}. Away from q_1, $T(G_{q_1}(Im\gamma))\big|_{Im\gamma}$ is spanned by $T(Im\gamma)$ and Y, so the X_i are in fact normal to $G_{q_1}(Im\gamma)$. $\square$

We can now prove

Theorem 4.0.12 *If $n > 2$, and g is a nonnegatively curved invariant metric on a $SU(2)$-manifold M^4 of type VII with principal isotropy subgroup Z_n, then (M^4, g) admits an $SU(2)$- equivariant Riemannian submersion to S^2 with totally geodesic fibers.*

Proof: Keeping the notation from the last few lemmas, $exp\big|_{D_l\nu G(q_1)} : D_l\nu G(q_1) \longrightarrow M^4 \setminus G(q_2)$ is a diffeomorphism since the $SU(2)$-geodesic segments given by $t \mapsto exp(tv)$, $v \in \nu G(q_1)$, $|v| = 1$, are $SU(2)$-minimizing on the interval $[0, l]$. The bundle projection $D_l(\nu G(q_1)) \longrightarrow G(q_1)$ therefore induces a C^∞ submersion $\tilde{\pi}_1 : M^4 \setminus G(q_2) \longrightarrow G(q_1)$. $\tilde{\pi}_1$ is $SU(2)$-equivariant, so in order to show that it is a Riemannian submersion, we need only check the Riemannian submersion property along $\gamma([0, l]) \subseteq M^4 \setminus G(q_2)$, where γ is the same as in the preceeding lemmas. Pick $t_0 \in (0, l)$ and $X(t_0) \in \nu_{\gamma(t_0)}\tilde{\pi}_1^{-1}(q_1) =$

$\nu_{\gamma(t_0)}(G_{q_1}\gamma([0,l))$. By the previous lemma, $X(t_0)$ extends to a Killing field which is parallel along $Im\gamma$ and normal to $G_{q_1}(\gamma([0,l)) = \bar{\pi}_1^{-1}(q_1)$. As before, call the restriction of this Killing field to $Im\gamma$, X. We have $|\bar{\pi}_{1*}X(t_0)| = |X(0)| = |X(t_0)|$ since $\bar{\pi}_1$ is $SU(2)$-equivariant. Hence $\bar{\pi}_{1*}$ is orthogonal on the fiber normals, and $\bar{\pi}_1$ is a Riemannian submersion.

In fact, $\bar{\pi}_1 : M^4 \setminus G(q_2) \longrightarrow G(q_1)$ extends to a C^∞ submersion $\pi : M^4 \longrightarrow G(q_1)$. To see this, observe that if $\bar{\pi}_2 : M^4 \setminus G(q_1) \longrightarrow G(q_2)$ is the Riemannian submersion obtained by imitating the construction of $\bar{\pi}_1$ but with $G(q_1)$ replaced by $G(q_2)$, then the fibers of $\bar{\pi}_2$ coincide with the fibers of $\bar{\pi}_1$ in $(M^4 \setminus G(q_2)) \cap (M^4 \setminus G(q_1)) = M^4 \setminus (G(q_1) \cup G(q_2))$. We can smoothly identify $G(q_1)$ and $G(q_2)$ as the quotient space of $M^4 \setminus (G(q_1) \cup G(q_2))$ by the same smooth relation, i.e. there is a diffeomorphism $\phi : G(q_2) \longrightarrow G(q_1)$ so that

$$\phi \circ \bar{\pi}_2\big|_{M^4\setminus(G(q_1)\cup G(q_2))} : M^4 \setminus (G(q_1) \cup G(q_2)) \longrightarrow G(q_1)$$

$$= \bar{\pi}_1\big|_{M^4\setminus(G(q_1)\cup G(q_2))}$$

The submersions $\bar{\pi}_1$ and $\phi \circ \bar{\pi}_2$ can then be glued along $M^4 \setminus (G(q_1) \cup G(q_2))$ to give a smooth Riemannian submersion $\pi : M^4 \longrightarrow G(q_1)$. $\square$

References

[1] J. Cheeger, *Some examples of manifolds of nonnegative curvature*, J. Differential Geometry **8** (1973), 623–628.

[2] J. Cheeger & D. Ebin, *Comparison theorems in Riemannian Geometry*, North-Holland, Amsterdam, 1975.

[3] M. H. Freedman, *The topology of four-dimensional manifolds*, J. Differential Geometry, **17** (1982) no. 3, 357–453.

[4] R. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geometry **17** (1982) no. 2, 255–306.

[5] S. Kobayashi, *Transformation groups in differential geometry*, Springer, New York, 1972.

[6] M. Gromov, *Curvature, Diameter, Betti numbers*, J. Differential Geometry.