

On Eschenburg's Habilitation on biquotient

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In 1984 Jost Eschenburg wrote his Habilitation on biquotients [E1]. This has been an important and influential paper which has laid the foundation for the theory of biquotients in Riemannian geometry, motivated by the Gromoll-Meyer paper from 1974 [GM] which showed that an exotic 7- sphere can be represented as a biquotient. The new examples of positive curvature which appeared in the Habilitation were later published in separate papers [E2],[E3]. But in addition to the fact that Eschenburg's Habilitation is not easily accessible, the fact that it was written in German has unfortunately prevented a general knowledge of its full content. This motivated me to give two lectures at the University of Pennsylvania at my secret seminar summarizing the content of the Habilitation. I especially wanted to describe explicitly the classification results and his tables on biquotients of equal rank in an easily accessible form. This is potentially of interest also outside of the subject of positive curvature. These notes do not contain any material that cannot be found in his Habilitation. A scanned copy of the Habilitation is available on my home page www.math.upenn.edu/~wziller/research.html

1. MAIN THEOREMS

Let G be a compact Lie group with left-invariant metric $\langle \cdot, \cdot \rangle$. Let K be the maximal subgroup of G such that $\langle \cdot, \cdot \rangle$ is right K -invariant. We call K the invariance group of $\langle \cdot, \cdot \rangle$. Let U be a closed subgroup of $G \times G$, and let U_L, U_R denote the projection of U onto the left and right factor of $G \times G$ respectively. We assume that $U_R \subset K$, i.e. that the metric is U_R -invariant. Then U acts isometrically on G via

$$(u_L, u_R) \cdot g = u_L g u_R^{-1} \quad , \quad (u_L, u_R) \in U.$$

This action is free if and only if, for all $g \in G$, $g \neq e$, we have $u_L \neq g u_R g^{-1}$. Notice also that the action is free if and only if $u_L \neq g u_R g^{-1}$ for any (u_L, u_R) in a maximal torus in U . In this case $\langle \cdot, \cdot \rangle$ induces a Riemannian metric on the quotient manifold $G//U$. The manifold $G//U$ is called a biquotient. It is sometimes advantageous to only assume that the action is free modulo an ineffective kernel, i.e. if $u_L = g u_R g^{-1}$ then $u_L = u_R$ lies in the center of G . In the case of $SU(n)$ it is also sometimes convenient to describe the biquotient as a quotient of $SU(n)$ by $U \subset U(n) \times U(n)$ where the first and second component have the same determinant (although this can always be rewritten as an ordinary biquotient of $U(n)$ as well).

We mention that Ochiai-Takahashi ('76) showed that for any simple Lie group G with left-invariant metric $\langle \cdot, \cdot \rangle$, the identity component of the isometry group is contained in $G \times G$, i.e. is of the form $G \times K$, for some $K \subset G$. For non-simple Lie groups, Ozeki showed that the same is true up to isometry. Thus any quotient of a Lie group with a left invariant metric by a group of isometries is of the above form.

We call a left invariant metric on G *torus invariant* if it is also right invariant under a maximal torus of G , i.e. $\text{rank } K = \text{rank } G$.

THEOREM 1.1 (Eschenburg). *If $M^n = G//U$ admits a torus invariant metric with positive curvature, then*

$$\text{rank}(G) = \begin{cases} \text{rank}(U) & \text{if } n \text{ is even,} \\ \text{rank}(U) + 1 & \text{if } n \text{ is odd.} \end{cases}$$

THEOREM 1.2 (Eschenburg). *Suppose G is simple and $G//U$ is even dimensional and admits a torus invariant metric with positive curvature. Then $G//U$ is diffeomorphic to a homogeneous space or $SU(3)//T^2$, where $T^2 = \{(diag(z, w, zw), diag(1, 1, z^2w^2)) \mid z, w \in S^1\}$.*

One easily sees that

$$S_{p,q}^1 = \{(diag(z^{p_1}, z^{p_2}, z^{p_3}), diag(z^{q_1}, z^{q_2}, z^{q_3})) \mid z \in S^1, \sum p_i = \sum q_i\}$$

acts freely on $SU(3)$ if and only if $(p_1 - q_{\sigma(1)}, p_2 - q_{\sigma(2)}) = 1$ for all $\sigma \in S_3$. The resulting biquotients $E_{p,q}^7 := SU(3)//S_{p,q}^1$ are known as Eschenburg spaces.

THEOREM 1.3 (Eschenburg). *Assume that G is semi-simple, $\text{rank}(G) = 2$, and $G//U$ is odd dimensional. If it admits a torus invariant metric with positive curvature, then $G//U$ is diffeomorphic to a homogeneous space or $E_{p,q}^7$.*

Note that the hypotheses of Theorem 1.3, together with Theorem 1.1, imply that $G \in \{S^3 \times S^3, SU(3), Sp(2), G_2\}$ and $U \in \{S^1, SO(3), SU(2)\}$.

As explained in Eschenburg's published papers, the manifold $SU(3)//T^2$ and the biquotients $E_{p,q}^7$ with $q_i \notin [\min\{p_j\}, \max\{p_j\}]$ for all $i = 1, 2, 3$, admit a metric with positive curvature.

Eschenburg's work was continued in odd dimensions in a Ph.D. Thesis of Bock in 1995. This paper was also written in German. In addition it was never published in any journal and we will hence shortly describe its main result.

Consider the embedding $Sp(2) \subset SU(4) \subset SU(5)$ given by

$$A \rightarrow \left(\begin{array}{cc|c} B & -\bar{C} & \\ C & \bar{B} & \\ \hline & & 1 \end{array} \right),$$

where $A = B + jC$, and $B, C \in M_2(\mathbb{C})$. Then let $Sp(2) \times S_p^1 \subset U(5) \times U(5)$ be the subgroup given by

$$\left\{ (diag(z^{p_1}, \dots, z^{p_5}), diag(A, z)) \mid z \in S^1, A \in Sp(2), p = \sum p_i \right\}.$$

One easily sees that $Sp(2) \cdot S_p^1 := (Sp(2) \times S_p^1)/\{\pm(I, e)\}$ acts freely on $SU(5)$ if and only if all of the p_i are odd and $(p_{\sigma(1)} + p_{\sigma(2)}, p_{\sigma(3)} + p_{\sigma(4)}) = 2$ for all $\sigma \in S_5$. The biquotients $B_p^{13} := SU(5)//Sp(2) \cdot S_p^1$ are known as Bazaikin spaces.

THEOREM 1.4 (Bock). *Suppose $M^{2n+1} = G//U$ admits a torus invariant metric with positive curvature, where $U = H \cdot H'$ with H of rank one, H' has no rank one factors, and H' acts only on one side. Then $G//U$ is diffeomorphic to a homogeneous space, $E_{p,q}^7$ or B_p^{13} .*

The strategy in Bock's thesis is different. He does not classify the tori T which act freely on G with $\text{rank } T = \text{rank } G - 1$ since they are too numerous. He uses the assumption that the group H' is a subgroup of G with $\text{rank } H' = \text{rank } G - 2$ and first classifies all such pairs (G, H') . He then examines which further rank 1 groups can act on both sides. He also uses along the way the 0-curvature criteria explained in the next section to simplify the discussion.

2. TORUS-INVARIANT METRICS

Let T be a maximal torus in G and $\langle \cdot, \cdot \rangle$ a torus-invariant metric. Recall that each real representation of a torus decomposes into 2-dimensional irreducible representations. In particular, this applies to the adjoint representation of T on \mathfrak{g} . Therefore \mathfrak{g} decomposes into the Lie algebra \mathfrak{t} of T and a sum of 2-dimensional representation modules of T , called *root spaces*. The differential of the representation of T on a root space E looks as follows: There is a linear form $r : \mathfrak{t} \rightarrow \mathbb{R}$ and a basis X, Y for E , such that for all $Z \in \mathfrak{t}$:

$$[Z, X] = -r(Z)Y \quad , \quad [Z, Y] = r(Z)X.$$

We denote the root space associated to each root $r : \mathfrak{t} \rightarrow \mathbb{R}$ by $E(r)$. Then on each $E(r)$, for all $Z \in \mathfrak{t}$

$$\text{ad}_Z = \begin{pmatrix} 0 & r(Z) \\ -r(Z) & 0 \end{pmatrix},$$

i.e. ad_Z is skew-symmetric.

The root spaces $E(r)$ are inequivalent as $\text{Ad}(T)$ -representations. Therefore, with respect to any torus-invariant metric, the root spaces $E(r)$ and \mathfrak{t} are pairwise orthogonal. Hence, the metric is arbitrary on \mathfrak{t} , and on $E(r)$ has the form $\alpha_r Q|_{E(r)}$, where Q is a bi-invariant metric on G and $0 < \alpha_r \in \mathbb{R}$.

Example: $G = SU(3)$. Then

$$\mathfrak{su}(3) = \begin{pmatrix} \cdot & \square_1 & \square_2 \\ & \cdot & \square_3 \\ & & \cdot \end{pmatrix},$$

where \square_i , $i = 1, 2, 3$, denote the root spaces, and $\mathfrak{t} = \{\text{diag}(a_1, a_2, a_3) \mid \sum a_i = 0\}$. Since $SU(3)$ is simple, a bi-invariant metric is unique up to a multiple and we set $Q(A, B) = -\frac{1}{2} \text{tr } AB$. Thus, letting Q be the Killing form, every torus-invariant metric $\langle \cdot, \cdot \rangle$ on $SU(3)$ has the form

$$\langle \cdot, \cdot \rangle = b + \alpha_1 Q|_{\square_1} + \alpha_2 Q|_{\square_2} + \alpha_3 Q|_{\square_3},$$

where b is an arbitrary metric on \mathfrak{t} .

3. CURVATURE OF TORUS-INVARIANT METRICS

The proof of Eschenburg's classification theorems consists of two parts. In the first part one needs to classify all biquotients $G//U$ with G simple and $\text{rank}(G) = \text{rank}(U)$. In the second part one needs to develop criteria for 0 curvature planes that can be applied to each case. We start with the second more geometric part.

Let $\langle \cdot, \cdot \rangle$ be a left invariant metric on G and Q a fixed biinvariant metric. We define the metric tensor $P: \mathfrak{g} \rightarrow \mathfrak{g}$ by $\langle X, Y \rangle = Q(X, P(Y))$ for $X, Y \in \mathfrak{g}$. We also denote by $\langle \cdot, \cdot \rangle$ the induced metric on $G//U$. We can identify the vertical space \mathcal{V}_g at $g \in G$ via left translations with $d(L_{g^{-1}})_*(\mathcal{V}_g)$, which for simplicity we again denote by \mathcal{V}_g .

THEOREM 3.1. *The following are sufficient conditions for a zero curvature plane of the metric $\langle \cdot, \cdot \rangle$ on $G//U$ at a point gU :*

- (N1) *There exists a P -invariant abelian subalgebra \mathfrak{a} and linearly independent $X, Y \in \mathfrak{a}$ which are perpendicular to \mathcal{V}_g ;*
- (N2) *There are P -invariant subspaces $W_1, W_2 \subset \mathfrak{g}$ with $[W_1, W_2] = 0$, and for some linearly independent vectors $X \in W_1, Y \in W_2$, perpendicular to \mathcal{V}_g , we have $[Y, P(Y)] \in W_2$;*
- (N3) *There is an $\text{Ad}(K)$ -invariant eigenspace V of P with $V \perp \mathfrak{u}_R$, and for some linearly independent vectors $X \in \mathfrak{k}$ and $Y \in V$, perpendicular to \mathcal{V}_g , we have $[P(X), Y] = 0$.*

Proof. We use the O'Neill formula:

$$\text{sec}_{G//U}(x, y) = \text{sec}_G(\bar{x}, \bar{y}) + \frac{3}{4} \| [X, Y]^\mathcal{V} \|^2,$$

where x, y are orthonormal horizontal vectors, \bar{x}, \bar{y} their horizontal lift, X, Y horizontal vector fields extending \bar{x}, \bar{y} , and $[X, Y]^\mathcal{V}$ denotes the vertical part of $[X, Y]$.

For the curvature of the left invariant metric $\text{sec}_G(\bar{x}, \bar{y})$ we use a formula of Püttmann:

$$\begin{aligned} \langle R(X, Y)Y, X \rangle &= \frac{1}{2}Q([PX, Y] + [X, PY], [X, Y]) - \frac{3}{4}Q(P[X, Y], [X, Y]) \\ &\quad + Q(B(X, Y), P^{-1}B(X, Y)) - Q(B(X, X), P^{-1}B(Y, Y)), \end{aligned}$$

where $B(X, Y) = \frac{1}{2}([X, PY] - [PX, Y])$.

For the O'Neill term $\| [\bar{X}, \bar{Y}]^\mathcal{V} \|^2$, one needs to develop a formula as well. Let $a, b \in \mathfrak{g}$ be left-invariant vector fields and A, B be horizontal vector fields (with respect to $G \rightarrow G//U$) such that $A(g) = a(g)$ and $B(g) = b(g)$. Define $z(a, b; g) := \|[A, B]^\mathcal{V}(g)\|$. Let \mathfrak{u} be the Lie algebra of U . The adjoint, $(\text{ad}_a)^*$, of ad_a with respect to the left-invariant metric $\langle \cdot, \cdot \rangle$ is given by $(\text{ad}_a)^* = -P^{-1} \circ \text{ad}_a \circ P$. If we define $\mathcal{L}(a, b) = (\text{ad}_a)^*(b) - (\text{ad}_b)^*(a) - [a, b]$, one shows that

$$z(a, b; g) = \max_{X \in \mathfrak{u}} \left| \frac{\langle \text{Ad}_{g^{-1}} X_L, \mathcal{L}(a, b) \rangle - \langle X_R, [a, b] \rangle}{|X^*(g)|} \right|,$$

where X^* is the action field of $X = (X_L, X_R) \in \mathfrak{u}$ on G .

Thus, if $a, b \in \mathfrak{g}$ span a horizontal 2-plane $\sigma_g \subset T_g G$ with $\text{sec}_G(a, b) = 0$ and $z(a, b; g) = 0$, then σ_g projects to a zero curvature plane at $g \cdot U \in G//U$. If $\langle \cdot, \cdot \rangle$ is bi-invariant, then σ_g has zero curvature if and only if $[a, b] = 0$.

Using all of the above, one now easily verifies the criteria (N1)-(N3). \square

4. EXAMPLES

In each of the following examples we assume that G is equipped with a torus-invariant metric, i.e. the invariance group K contains a maximal torus.

Example 1: There is a zero curvature plane at every point of $\mathrm{Sp}(2)//\mathbb{S}^1$ for any circle $\mathbb{S}^1 \subset \mathrm{Sp}(2) \times \mathrm{Sp}(2)$. To see this, consider the following subalgebras of $\mathfrak{sp}(2)$:

$$\begin{aligned} \mathfrak{t} &= \{\mathrm{diag}(i\alpha, i\beta) \mid \alpha, \beta \in \mathbb{R}\}, \\ V_1 &= \{\mathrm{diag}(a_1j + a_2k, 0) \mid a_1, a_2 \in \mathbb{R}\}, \text{ and} \\ V_2 &= \{\mathrm{diag}(0, b_1j + b_2k) \mid b_1, b_2 \in \mathbb{R}\}. \end{aligned}$$

V_1 and V_2 are root spaces, are hence P -invariant, and satisfy $V_1 \perp V_2$ and $[V_1, V_2] = 0$. Also, $[V_i, V_i] \subset V_i$, $i = 1, 2$, since V_1 and V_2 are subalgebras. For all $g \in \mathrm{Sp}(2)$, \mathcal{V}_g is one-dimensional and so we may find horizontal $X \in V_1$ and $Y \in V_2$, and $X \perp Y$. Thus by applying (N2), we see that $\mathrm{span}\{X, Y\}$ projects to a zero curvature plane at $g \cdot \mathbb{S}^1 \in \mathrm{Sp}(2)//\mathbb{S}^1$.

Example 2: Eschenburg showed that on $E_{p,q}^7$ there exists a special metric which has positive curvature if $q_i \notin [\min\{p_j\}, \max\{p_j\}]$ for all $i = 1, 2, 3$. We will now see that if $q_i \in [\min\{p_j\}, \max\{p_j\}]$, then $E_{p,q}^7$ has a zero-curvature plane at some point for *any* torus invariant metric. Since the other cases are similar, we assume that $q_3 \in [\min\{p_j\}, \max\{p_j\}]$. Using the same method that Eschenburg used to construct a metric of positive curvature in his examples, one sees that there is a $g \in \mathrm{SU}(3)$ such that $Q(\mathrm{Ad}_{g^{-1}}X_L - X_R, Y_3) = 0$, where $X = (X_L, X_R) \in \mathfrak{u}$, and $Y_3 = i \mathrm{diag}(1, 1, -2)$. Define $Y := P^{-1}(Y_3)$.

Let

$$V_1 := \left\{ \left(\begin{array}{cc|c} 0 & x & 0 \\ -\bar{x} & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \middle| x \in \mathbb{C} \right\}.$$

Then, since V_1 is two-dimensional, we can find $X \in V_1$ such that X is horizontal at g . Since V_1 is a root space, we have $\mathrm{Ad}(K)V_1 = \mathrm{Ad}(T)V_1 \subset V_1$. We also have $0 = [X, Y_3] = [X, P(Y)]$, and so we may apply (N3) to show that $E_{p,q}^7$ has a zero-curvature plane at $g \cdot \mathbb{S}_{p,q}^1$.

Example 3: We will now show that the Gromoll-Meyer sphere $\Sigma^7 := \mathrm{Sp}(2)//\mathrm{Sp}(1)$ has positive curvature at a point, but does not have positive curvature everywhere. Here we take $\mathrm{Sp}(1) = \{(\mathrm{diag}(q, q), \mathrm{diag}(q, 1)) \mid q \in \mathrm{Sp}(1)\} \subset \mathrm{Sp}(2) \times \mathrm{Sp}(2)$. Consider the identity $I \in \mathrm{Sp}(2)$. We claim that Σ^7 has positive curvature at $I \cdot \mathrm{Sp}(1)$. The vertical subspace at I is given by

$$\mathcal{V}_I = \left\{ \left(\begin{array}{cc} 0 & 0 \\ 0 & x \end{array} \right) \middle| x \in \mathrm{Im}(\mathbb{H}) \right\} \subset \mathfrak{sp}(2).$$

Hence the horizontal subspace at I is given by

$$\mathcal{H}_I = \left\{ \left(\begin{array}{cc} y & v \\ -\bar{v} & 0 \end{array} \right) \middle| y \in \mathrm{Im}(\mathbb{H}), v \in \mathbb{H} \right\} \subset \mathfrak{sp}(2),$$

which coincides with the horizontal space \mathfrak{H}_I at I of the homogeneous space $\mathbb{S}^7 = \mathrm{Sp}(2)/\mathrm{Sp}(1)$, where in this case $\mathrm{Sp}(1) = \{\mathrm{diag}(1, q) \mid q \in \mathrm{Sp}(1)\} \subset \mathrm{Sp}(2)$. Now, since \mathbb{S}^7 has positive curvature, any vectors $X, Y \in \mathfrak{H}_I$ such that $[X, Y] = 0$ must therefore be linearly dependent. Hence there are no horizontal zero-curvature planes at I , and so Σ^7 , with the metric induced by a biinvariant metric, has non negative curvature and has positive curvature

at $I \cdot \text{Sp}(1)$.

We now show that for any U_R -invariant metric on $\text{Sp}(2)$, Σ^7 has a plane of zero-curvature at some point, where $U_R = \{\text{diag}(q, 1) \mid q \in \text{Sp}(1)\}$. Consider the subspaces $W_1, W_2, W_3 \subset \mathfrak{sp}(2)$, where

$$\begin{aligned} W_1 &:= \left\{ \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \mid x \in \text{Im}(\mathbb{H}) \right\}, \\ W_2 &:= \left\{ \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix} \mid y \in \text{Im}(\mathbb{H}) \right\}, \text{ and} \\ W_3 &:= \left\{ \begin{pmatrix} 0 & v \\ -\bar{v} & 0 \end{pmatrix} \mid v \in \mathbb{H} \right\}. \end{aligned}$$

$\text{Ad}(U_R)$ acts on W_1, W_2 and W_3 by $\{x \rightarrow qxq^{-1}\}, \{\text{id}\}$ and $\{v \rightarrow qv\}$ respectively. Then W_1, W_2 and W_3 are clearly inequivalent $\text{Ad}(U_R)$ -representations, and by Schur's Lemma, are pairwise orthogonal. We remark that the metric on W_2 can be arbitrary, since $\text{Ad}(U_R)$ acts trivially.

The vertical subspace at $g \in \text{Sp}(2)$ is given by

$$\mathcal{V}_g = \left\{ \text{Ad}_{g^{-1}} \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} - \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \mid x \in \text{Im}(\mathbb{H}) \right\}.$$

Let $g = \frac{1}{2} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$. Then

$$\mathcal{V}_g = \text{span} \left\{ \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} j & k \\ k & 0 \end{pmatrix}, \begin{pmatrix} -k & j \\ j & 0 \end{pmatrix} \right\}.$$

Let $X = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix} \in W_1$ and $Y = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \in W_2$, where $b \perp i$. Therefore $X, Y \in \mathcal{H}_g$. Applying (N2) we see that $\sigma = \text{span}\{X, Y\}$ projects to a zero-curvature plane at $g \cdot \text{Sp}(1) \in \Sigma^7$.

Example 4: For any U_R -invariant metric on $\text{SO}(2n+1)$, the biquotient $M := \Delta \text{SO}(2) \backslash \text{SO}(2n+1) / \text{SO}(2n-1)$ has a zero-curvature plane at every point. Here

$$\Delta \text{SO}(2) = \left\{ \begin{pmatrix} A & & & \\ & \ddots & & \\ & & A & \\ & & & 1 \end{pmatrix} \mid A \in \text{SO}(2) \right\} \subset \text{SO}(2n+1).$$

Note that $\text{rank}(\text{SO}(2n+1)) = \text{rank}(\Delta \text{SO}(2) \times \text{SO}(2n-1))$, and also that M is, in fact, the quotient of the unit tangent bundle, $T_1\mathbb{S}^n$, of \mathbb{S}^n , and the action of $\Delta \text{SO}(2)$ on $T_1\mathbb{S}^n$ is the geodesic flow.

Let $V, W = \mathbb{R}^{2n-1}$. Then we may write

$$\mathfrak{so}(2n+1) = \left\{ \left(\begin{array}{c|cc} \mathfrak{so}(2n-1) & x & y \\ \hline -x^t & 0 & a \\ -y^t & -a & 0 \end{array} \right) \middle| x \in V, y \in W, a \in \mathbb{R} \right\}.$$

Then $\mathfrak{so}(2n-1)^\perp = V \oplus W \oplus \mathbb{R}$. Now $\text{Ad}(SO(2n-1))$ acts on $\mathfrak{so}(2n-1)^\perp$ via

$$\text{Ad}_A(x, y, a) = (Ax, Ay, a).$$

Since $V \perp W$, we may choose g such that $\text{Ad}_g V \perp \text{Ad}_g W$. Now, all $\text{Ad}(SO(2n-1))$ -invariant subspaces are of the form $\text{Ad}_g V$, for some g of the form

$$\left(\begin{array}{c|cc} \text{I} & & \\ \hline & a & b \\ & c & d \end{array} \right) \in \text{SO}(2).$$

Noting that $[V, W] = 0$, we now choose $X \in \text{Ad}_g V$, $Y \in \text{Ad}_g W$ such that $[X, Y] = 0$. Note also that $P(\text{Ad}_g V) \subset \text{Ad}_g V$ and $P(\text{Ad}_g W) \subset \text{Ad}_g W$.

Then for all $h \in \text{SO}(2n+1)$ we use the fact that $\Delta \text{SO}(2)$ is one-dimensional to choose X, Y as above such that $X, Y \perp \mathcal{V}_h := \text{Ad}_{h^{-1}}(\Delta \mathfrak{so}(2)) - \mathfrak{so}(2n-1)$.

We may now apply (N1) to see that there is thus a zero-curvature plane at every point of $\Delta \text{SO}(2) \setminus \text{SO}(2n+1) / \text{SO}(2n-1)$.

Note that Wilking ('02) equipped $\Delta \text{SO}(2) \setminus \text{SO}(2n+1) / \text{SO}(2n-1)$ with a metric of almost positive curvature, i.e. a metric which has positive curvature on an open dense set of points.

5. CLASSIFICATION OF BIQUOTIENT ACTIONS BY A MAXIMAL TORUS

A major achievement in Eschenburg's Habilitation is the classification of all biquotients $G//U$ where G is simple and $\text{rank}(G) = \text{rank}(U)$. This is based on first classifying all maximal tori which act freely on G , which we will describe now.

We start with free biquotient actions on $\text{SU}(n)$.

THEOREM 5.1. *For $n \geq 3$ a torus $\mathbb{T}^{n-1} = \langle z, w_1, \dots, w_{n-2} \rangle$ acts freely on $\text{SU}(n)$ if and only if it either acts on one side, or \mathbb{T}^{n-1} is conjugate to $\text{S}_{1,\ell}$ or $\text{S}_{2,\ell}$ for some $1 \leq \ell \leq \frac{n}{2}$, where*

$$\text{S}_{1,\ell}^{n-1} = \{ \text{diag}((z^2, \dots, z^2, 1, \dots, 1); \text{diag}((z\bar{w}_1 \dots \bar{w}_{n-2}), z^2 w_1, \dots, z^2 w_{\ell-1}, w_\ell, \dots, w_{n-2}, z)) \}$$

$$\text{S}_{2,\ell}^{n-1} = \{ \text{diag}(1, \dots, 1, z^2); \text{diag}((z\bar{w}_1 \dots \bar{w}_{\ell-1}), w_1, \dots, w_{n-2}, (z\bar{w}_\ell \dots \bar{w}_{n-2})) \}$$

The actions of $\text{S}_{1,\ell}$ and $\text{S}_{2,\ell}$ are equivalent if and only if $\ell = 1$.

Remark 1. Recall that we must have $\det(u_L) = \det(u_R)$, where $(u_L, u_R) \in U = S_{i,\ell}$, $i = 1, 2$. Thus in $S_{1,\ell}$ there are ℓ copies of z^2 on the left-hand side.

Remark 2. If $n = 2m$, we may rewrite the actions in Theorem 5.1 with $\ell = m$ as

$$\begin{aligned} S_{1,m}^{2m-1} &= \{ \text{diag}((z, \dots, z, \bar{z}, \dots, \bar{z}); \text{diag}((\bar{w}_1 \dots \bar{w}_{n-2}), w_1, \dots, w_{n-2}, 1)) \} \\ S_{2,m}^{2m-1} &= \{ \text{diag}(z, \dots, z, z^{n-1}); \text{diag}((\bar{w}_1 \dots \bar{w}_{m-2}), w_1, \dots, w_{n-2}, (\bar{w}_m \dots \bar{w}_{n-2})) \} \end{aligned}$$

Remark 3. Note that in the biquotient actions on $SU(n)$ there is only one S^1 which acts on both sides of $SU(n)$.

THEOREM 5.2. *For $n \geq 3$ an n -torus $T^n = \langle z, w_1, \dots, w_{n-1} \rangle$ acts freely on $\text{Sp}(n)$ if and only if it either acts on one side, or T^n is conjugate to P_1^n or P_2^n , where*

$$\begin{aligned} P_1^n &= \{ \text{diag}((1, \dots, 1, z); \text{diag}(w_1, \dots, w_{n-1}, (\bar{w}_1 \dots \bar{w}_{n-1}))) \} \\ P_2^n &= \{ \text{diag}(z, \dots, z); \text{diag}(w_1, \dots, w_{n-1}, 1) \} \end{aligned}$$

The actions of P_1^n and P_2^n are equivalent if and only if $n = 2$.

If we consider the usual embeddings

$$U(n) \subset \text{SO}(2n) \subset \text{SO}(2n+1) \quad , \quad U(n) \subset \text{Sp}(n)$$

for $n \geq 2$, we see that a maximal torus in $U(n)$ can also be viewed as a maximal torus in $\text{SO}(2n)$, $\text{SO}(2n+1)$ or $\text{Sp}(n)$. Thus free biquotient actions on $\text{Sp}(n)$ give rise to free biquotient actions on $\text{SO}(2n)$ and $\text{SO}(2n+1)$, and vice versa. We do not consider the group $\text{SO}(4)$ since it is not simple.

We record the following special cases:

COROLLARY. *For the rank 2 groups we have:*

(a) *The only 2-torus acting freely on $SU(3)$ on both sides is:*

$$S_{1,1}^2 = S_{2,1}^2 = \{ \text{diag}(1, 1, z^2 w^2); \text{diag}(z, w, zw) \}$$

(b) *The only 2-torus acting freely on $\text{Sp}(2)$ on both sides is:*

$$P_1^2 = P_2^2 = \{ \text{diag}(z, z); \text{diag}(w, 1) \}$$

Remark. Due to the isomorphism $\text{Spin}(6) = \text{SU}(4)$, we have, besides the free actions of P_1^3, P_2^3 on $\text{SO}(6)$, a third torus acting freely which can be written as:

$$P_3^3 = \{ \text{diag}(z, z, z); \text{diag}(zw_1, w_2, \bar{w}_1 \bar{w}_2) \}$$

Finally, for the exceptional Lie groups, we have:

THEOREM 5.3. *The exceptional Lie groups G_2, F_4, E_6, E_7, E_8 admit no free two-sided torus actions of maximal rank.*

6. CLASSIFICATION OF MAXIMAL BIQUOTIENT ACTIONS OF MAXIMAL RANK

We now describe the classification when U is not abelian. Note that if a maximal torus T acts freely on G , then any extension U of T with $\text{rank}(U) = \dim T$ will also act freely on G . Eschenburg classified all such U which are maximal among these extensions, i.e. all $U_{max} \subset G \times G$ such that if U is another extension of T , then U is contained in some U_{max} . Given such a U_{max} and a torus $T \subset U_{max}$, maximal in U_{max} , it is a simple exercise to list all the extensions U with $T \subset U \subset U_{max}$ by using the Borel Siebenthal classification of maximal rank subgroups. For example, if T^{n-1} is the usual maximal torus in $SU(n)$, then extensions U are given by

$$T^{n-1} \subset S(U(n_1) \times \cdots \times U(n_k)) \subset SU(n),$$

where $\sum n_i = n$. Similarly, for T^n the usual maximal torus in $Sp(n)$, extensions U are given by

$$T^n \subset U_1 \times \cdots \times U_k \subset Sp(n),$$

where $U_i = Sp(n_i)$ or $U(n_i)$, and $\sum n_i = n$.

Finally, if T^n is the usual maximal torus in $SO(2n)$ or $SO(2n+1)$, extensions U are given by

$$T^n \subset U_1 \times \cdots \times U_k \subset SO(n),$$

where $U_i = SO(n_i)$ or $U(n_i/2)$, and $\sum n_i = n$.

We break up the description of all such maximal U into those where the quotient is diffeomorphic to a rank one symmetric space in Table A, and those which are not in Table B.

In these Tables, $U = U_1 \times U_2$ where U_1 is a rank one factor which, except in case 1 and 9, is embedded only on the left. In all cases U_2 acts only on the right.

For case 1, U is a semidirect product $S^1 \ltimes SU(n-1)$ where $SU(n-1) = \{\text{diag}(A, 1) \mid A \in SU(n-1)\}$ acts only on the right and

$$S^1 = \{ \text{diag}(z^2, \dots, z^2, 1, \dots, 1); \text{diag}(z, z^2, \dots, z^2, 1, \dots, 1, z) \}$$

on both sides. Indeed, this circle subgroup $S^1 \subset G \times G$ clearly normalizes $(\{e\} \times \text{diag}(A, 1))$. Similarly for case 9, where

$$S^1 = \{ \text{diag}(1, \dots, 1, z^2); \text{diag}(z, 1, \dots, 1, z) \}.$$

In case 10 on the other hand, the circle acts only on the left as $\text{diag}(z, \dots, z, z^{n-1})$.

The diagonal subgroup: $\Delta SO(2) \subset SO(2n)$ and $\Delta SU(2) \subset SO(4n)$ act as Hopf actions on $\mathbb{S}^{2n-1} = SO(2n)/SO(2n-1)$ respectively $\mathbb{S}^{4n-1} = SO(4n)/SO(4n-1)$. Furthermore, $\Delta SO(2) \subset SO(2n+1)$ and $\Delta SU(2) \subset SO(4n+1)$ are obtained via first embedding into $SO(2n)$ respectively $SO(4n)$.

In case 11,12, 16,17 the embedding of U_1 is the standard block embedding.

	G	n	T_U	$U = U_1 \times U_2$	$G//U$
1	$SU(n)$	$n \geq 5$	$S_{1,\ell}, 2 \leq \ell < \frac{n}{2}$	$S_\ell^1 \times SU(n-1)$	$\mathbb{C}P^{n-1}$
2	$SU(2n)$	$n \geq 2$	$S_{1,n}$	$\Delta SU(2) \times SU(2n-1)$	$\mathbb{H}P^{n-1}$
3	$Spin(7)$		P_1^3	$Spin(3) \times G_2$	S^4
4	$Spin(8)$		P_1^4	$Spin(3) \times Spin(7)'$	S^4
5	$Spin(9)$		P_1^4	$Spin(3) \times Spin(7)'$	$\mathbb{H}P^3$
6	$SO(2n)$	$n \geq 3$	P_2^n	$\Delta SO(2) \times SO(2n-1)$	$\mathbb{C}P^{n-1}$
7	$SO(4n)$		P_2^{2n}	$\Delta SU(2) \times SO(4n-1)$	$\mathbb{H}P^{n-1}$
8	$Sp(n)$	$n \geq 2$	P_2^n	$\Delta Sp(1) \times Sp(n-1)$	$\mathbb{H}P^{n-1}$

TABLE A. Maximal rank free actions such that $G//U$ is diffeomorphic to a compact rank one symmetric space.

	G	n	T_U	$U = U_1 \times U_2$
9	$SU(n)$	$n \geq 5$	$S_{2,\ell}, 2 \leq \ell < \frac{n}{2}$	$S^1 \times SU(\ell) SU(n-\ell)$
10	$SU(2n)$	$n \geq 2$	$S_{2,n}$	$S^1 \times SU(n) SU(n)$
11	$SO(2n)$	$n \geq 5$	P_1^n	$SO(3) \times SU(n)$
12	$SO(2n+1)$	$n \geq 5$	P_1^n	$SO(3) \times SU(n)$
13	$SO(2n+1)$	$n \geq 3$	P_2^n	$\Delta SO(2) \times SO(2n-1)$
14	$SO(2n)$	$2n = p + q \geq 2,$ p, q odd	P_2^n	$\Delta SO(2) \times SO(p) SO(q)$
15	$SO(4n+1)$	$n \geq 2$	P_2^{2n}	$\Delta SU(2) \times SO(4n-1)$
16	$Sp(n)$	$n \geq 3$	P_1^n	$Sp(1) \times SU(n)$
17	$Sp(4)$		P_1^4	$Sp(1) \times SU(2)^3$

TABLE B. Maximal rank free actions such that $G//U$ is not diffeomorphic to a compact rank one symmetric space.

The embedding of U_2 , which only acts on the right, is the standard block embedding when U_2 is a classical group. In case 3 it is the standard embedding of G_2 in $SO(7)$, lifted to $Spin(7)$ and in case 4 and 5 the spin embedding.

The only other embedding that needs to be described is the embedding of $U_2 = SU(2)^3 \subset Sp(4)$ in case 17. For this we consider the representation $SU(2)^3 \subset SU(8)$

given by the exterior tensor product of the tautological 2 dimensional representation of $SU(2)$ on each factor. Since this representation is symplectic, the image lies in $Sp(4)$.

We point out that entry 14 was missing in its full generality in [E1], as was observed in [EKS]. Due to this fact it is not yet certain that his classification is complete. This should be settled in the forthcoming English translation by Catherine Searle and Jost Eschenburg.

We finally mention his classification in the rank 2 case of all biquotients, not just the equal rank ones:

THEOREM 6.1. *If G is a simple Lie group of rank 2 and U a rank one group acting freely as a biquotient, then it is either a homogeneous space, or U is the circle action on $SU(3)$ whose quotient is the Eschenburg spaces E_{pq}^7 , or the Gromoll-Meyer biquotient action of $Sp(1)$ on $Sp(2)$ (or its subgroup $S^1 \subset Sp(1)$), or a biquotient action of $SU(2)$ (respectively $S^1 \subset SU(2)$) on G_2 . The latter acts via the index 3 three dimensional subgroup on the left, and the index 4 three dimensional one on the right.*

The biquotient $Sp(2)//Sp(1)$ is the famous Gromoll Meyer sphere [GM], which is homeomorphic but not diffeomorphic to a sphere. In [KZ] it was shown that $G_2//SU(2)$ is homeomorphic to T_1S^6 , but it is not known if it diffeomorphic to it or not. In the case of $Sp(2)//S^1$ and $G_2//S^1$ we do not know which 2-sphere bundle over the respective $SU(2)$ biquotient it is.

In [KS],[K2] a diffeomorphism classification was given of (almost all) Eschenburg spaces $E_{p,q}$ in terms of number theoretic sums. This was used in [CEZ] to study various homeomorphism and diffeomorphism properties of these manifolds. See also [AMP1, AMP2, E4, K1, Sh] for other topological properties.

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