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ON THE ISOTROPY REPRESENTATION OF A SYMMETRIC SPACE

It has been observed previoulsly that there is a close connnection between the isotropy representation of irreducible symmetric spaces and the theory of isotropy irreducible homogeneous spaces. In this note we will observe that there also exists such a close connection between the description of all maximal subgroups of classical compact simple Lie groups and the isotropy representation of irreducible symmetric spaces. It is our goal give to a conceptional proof of these relationships.

We first set some terminology for symmetric spaces. Let G/H be a simply connected, n-dimensional, irreducible Riemannian symmetric space of compact type. Then the isotropy representation of H on $T_{(H)}\, G/H$ gives a representation $\pi : H \to SO\,(n)$. We differentiate between three types of symmetric spaces:

1) G/H is hermitian symmetric and hence $H = H' \cdot U\,(1)$ with H' semi-simple. $\pi_{\mathbb{C}} = \pi_\lambda + \pi_\lambda{}^*$ where $\mathbb{C}$ denotes complexification and π_λ is the irreducible complex representation with dominant weight λ . We can write $\pi_\lambda = \pi_{\lambda'} \otimes \overset{1}{t}$ where $\overset{1}{t}$ is the effective one dimensional representation of $U\,(1)$.

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* The first author was partially supported by NSERC grant number U0257. The second author was partially supported by an Alfred P. Sloan Research Fellowship and by a grant from the National Science Foundation.

2) G/H is quaternionic symmetric iff $H = H' \cdot \mathrm{Sp}(1)$ with H' semi-simple and $\pi_{\mathbb{C}} = \pi_\lambda = \pi_{\lambda'} \otimes \overset{1}{o}$ where $\pi_{\lambda'}$ is symplectic and $\overset{1}{o}$ is the two dimensional representation of $Sp(1)$.

3) If G/H is such that H is simple of if G/H is a Grassmannian over $\mathbb{R}$ or $\mathbb{H}$, then we will call G/H "real" symmetric. Again we have $\pi_{\mathbb{C}} = \pi_\lambda$.

Every irreducible symmetric space belongs to one of these classes, but a given symmetric space can belong to two of them at the same time.

Isotropy irreducible homogeneous spaces

A homogeneous space A/B, where A and B are not necessarily connected Lie groups, is called isotropy irreducible if the isotropy representation of B on $T_{(B)} A/B$ is irreducible (over $\mathbb{R}$) and is called strongly isotropy irreducible if B_0, the connected identity component of B, acts irreducible on $T_{(B)} A/B$. Examples of isotropy irreducible homogeneous spaces which are not strongly isotropy irreducible are $M \times M \times \ldots \times M$ where M is a strongly isotropy irreducible homogeneous space or $W \backslash G/T$ where G is a compact simple Lie group with all roots of the same lengths, T a maximal torus and W the Weyl group of G.

In 1968 J. Wolf [W] classified strongly isotropy irreducible homogeneous spaces A/B which are not symmetric. In particular he showed that A must be a compact simple Lie group. After this classification, C.T.C. Wall observed (see [W], p. 147) that, in the case where A is a classical group $SO(n)$, $SU(n)$, or $Sp(n)$, there is a close connection between these strongly isotropy irreducible homogeneous spaces and the isotropy representation of a symmetric space. Roughly speaking, if G/H is an irreducible symmetric space of dimension n with isotropy representation $\pi : H \to SO(n)$ then either $SO(n)/\pi(H)$ is strongly isotropy irreducible or there exists a Lie group L with $\pi(H) \subsetneq L \subsetneq SO(n)$ and $L/\pi(H)$ is strongly isotropy irreducible. The possibilities for L are $U\left(\frac{n}{2}\right)$ and $Sp\left(\frac{n}{4}\right) \cdot Sp(1)$ and after $L/\pi(H)$ is made effective, we obtain quotients of $SU\left(\frac{n}{2}\right)$ and

$Sp\left(\frac{n}{4}\right)$. More precisely:

a) If G/H is hermitian symmetric, then $\pi(H) \subset U\left(\frac{n}{2}\right) \subset SO(n)$ and $U\left(\frac{n}{2}\right)/\pi(H) = SU\left(\frac{n}{2}\right)/\pi_{\lambda'}(H')$ is strongly isotropy irreducible. Conversely, if $SU(n)/\pi(H)$ is strongly isotropy irreducible but not symmetric, then the representation $\pi \otimes \overset{1}{t} + \pi^* \otimes \overset{-1}{t}$ of $H \cdot U(1)$ is the isotropy representation of a hermitian symmetric space.

b) If G/H is quaternionic symmetric, then $\pi(H) \subset Sp\left(\frac{n}{4}\right) \cdot Sp(1) \subset$ $\subset SO(n)$ and $Sp\left(\frac{n}{4}\right) \cdot Sp(1)/\pi(H) = Sp\left(\frac{n}{4}\right)/\pi_{\lambda'}(H')$ is strongly isotropy irreducible. Conversely if $Sp(n)/\pi(H)$ is strongly isotropy irreducible but not symmetric, then the representation $\pi \otimes \overset{1}{o}$ of $H \cdot Sp(1)$ is the isotropy representation of a quaternionic symmetric space.

c) If G/H is real symmetric, but not a real or quaternionic Grassmannian (or equivalently H is simple), then $SO(n)/\pi(H)$ is strongly isotropy irreducible. The Grassmannian $Sp(n+1)/Sp(n) \cdot Sp(1)$ also gives rise to a strongly isotropy irreducible homogeneous space $SO(4n)/Sp(n) \cdot Sp(1)$. Conversely, if $SO(n)/\pi(H)$ is strongly isotropy irreducible but not $SO(7)/G_2$, then π is the isotropy representation of a real symmetric space.

But a word of caution has to be added:

a)′ $P^n\mathbb{C}$ gives rise to a trivial homogeneous space and $SO(n+$ $+2)/SO(n) \cdot SO(2)$ gives rise to $SU(n)/SO(n)$ which is symmetric.

b)′ $P^n\mathbb{H}$ gives rise to a trivial homogeneous space and $SU(n+$ $+2)/S(U(n) \cdot U(2))$ gives rise to $Sp(n)/U(n)$ which is symmetric. One also has to take care not to forget $SO(n+4)/SO(n) \cdot SO(4)$ which is quaternionic symmetric and gives rise to $Sp(n)/SO(n) \cdot Sp(1)$.

c)′ S^n gives rise to a trivial homogeneous space.

All other symmetric spaces give rise to non-trivial non-symmetric strongly isotropy irreducible homogeneous spaces. A particularly nice family comes

from the real symmetric space $H \times H/\Delta H$ where H is a compact simple Lie group and gives rise to the strongly isotropy irreducible space $SO(\dim H)/Ad(H)$.

One should remark that the strongly isotropy irreducible spaces $SO(4n)/Sp(n) \cdot Sp(1)$ and $Sp(n)/SO(n) \cdot Sp(1)$ were not included in Wolf's list, see [W2].

The description of the above relationship differs only slightly from the one given by C.T.C. Wall. So far, the only proof of this relationship consists of comparing the classification of symmetric spaces with Wolf's classification of strongly isotropy irreducible homogeneous spaces. The goal of our discussion will be to give a conceptional proof, which will also explain the exceptions in the above description.

Maximal subgroups of classifical Lie groups.

In 1952 E. Dynkin [D] classified all maximal connected subgroups of classical compact simple Lie groups. If K is either $SO(n)$, $SU(n)$, or $Sp(n)$ and $H \subsetneq K$ is a maximal connected subgroup, then the inclusion of H in K can be described by a representation π which, depending on K, is orthogonal unitary or symplectic. One easily reduces this problem to the case where H is a simple Lie group and where the representation $\pi_{\mathbb{C}}$ is irreducible. Dynkin proved the remarkable theorem that conversely, if $\pi_{\mathbb{C}}$ is an irreducible representation of a simple Lie group H , then $\pi(H)$ is in fact a maximal subgroup of $SO(n)$, $SU(n)$, resp. $Sp(n)$, except for a short list of representations, see [D] , Table 1, p. 364. Dynkin's proof proceeds by a case by case classification.

As far as we know, no conceptual description of these exceptional representations has ever been given. In this note we will again be able to explain them in terms of isotropy representations of symmetric spaces. To do this let us first describe some easy reductions in Dynkin's classification. If H is not maximal, there exists a Lie group L with $H \subsetneq L \subsetneq K$ and we can describe the inclusion $L \subset K$ by a representation π' . Then π' is an irreducible representation of L which, when restricted to H , remains irreducible (since it equals π) . Conversely, if π' is an irreducible representation of

L and if there exists a subgroup $H \subset L$ such that $\pi = \pi' | H$ is irreducible and such that π and π' have the same type, i.e. are both orthogonal, both symplectic, or both unitary, then (H, π) will be among the exceptions mentioned before. One quickly observes that the only difficult case is where L is a simple Lie group and if L is exceptional, only certain representations of G_2 can occur. If L is a classical simple Lie group, one shows that the only possible representations π' are:

$$SO(n) \quad , \quad \pi' = \Delta^{\pm} \text{ (spin representations)}$$

$$SU(n) \quad , \quad \pi' = \Lambda^k \mu_n \quad , \quad k \geqslant 2$$

$$Sp(n) \quad , \quad \pi' = \sigma_k = \circ\!-\!\circ\!-\!\ldots\!-\!\overset{1}{\circ}\!-\!\ldots\!-\!\circ\!=\!\circ \quad , \quad k \geqslant 2$$

$$\text{where} \quad \sigma_2 = \Lambda^2 \nu_{2n} - id \ .$$

Here μ_n and ν_{2n} are the natural n and $2n$ dimensional representations of $SU(n)$ and $Sp(n)$.

If we describe $H \subset L$ by the representation π , the problem now becomes to classify those orthogonal representations π such that $\Delta^{\pm} \circ \pi$ is irreducible, those unitary representations π such that $\Lambda^k \pi$ is irreducible for some $k \geqslant 2$ and those symplectic representations π such that $\sigma_k \circ \pi$ is irreducible for some $k \geqslant 2$. The result of this classification is contained in Table 6, 11, 15, 16, 17 in $[D]$, p. 368-372, and one observes

a) If $\Lambda^k \pi$ is irreducible, then $\Lambda^2 \pi$ is irreducible, but not necessarily conversely. $\Lambda^2 \pi$ is irreducible iff the representation $\pi \otimes \overset{1}{t} + \pi^* \otimes \overset{-1}{t}$ of $H \cdot U(1)$ is the isotropy representation of an irreducible hermitian symmetric space $G/H \cdot U(1)$ with H simple.

b) If $\sigma_k \circ \pi$ is irreducible, then $\sigma_2 \circ \pi = \Lambda^2 \pi - id$ is irreducible but not necessarily conversely. $\Lambda^2 \pi - id$ is irreducible iff the representation $\pi \otimes \overset{1}{o}$ of $H \cdot Sp(1)$ is the isotropy representation of an irreducible quaternionic symmetric space $G/H \cdot Sp(1)$ with H simple

c) If $\Delta^{\pm} \circ \pi$ is irreducible, then π is the isotropy representation of a real symmetric space G/H with H simple, but not conversely.

Hence all the irreducible representations which give rise to non maximal subgroups of classical Lie groups can be directly described in terms of the isotropy representation of symmetric spaces. Again, our goal will be to give a conceptual proof of this relationship that does not contain any case by case arguments.

Symmetric Spaces.

We will now study which irreducible representations can be isotropy representations of symmetric spaces.

Let G/H be an irreducible symmetric space with isotropy representation π. Then $\pi_{\mathbb{C}} = \pi_\lambda$ or $\pi_{\mathbb{C}} = \pi_\lambda + \pi_\lambda^*$. On $\mathfrak{h}$ we have a natural inner product $(\ ,\ ) = -B_G|\mathfrak{h}$ where B_G is the Killing form of G.

Theorem 1: If G/H is an irreducible symmetric space, then

a) For every simple root α of H with $(\lambda, \alpha) \neq 0$ we have $(\lambda, \lambda) = 2(\lambda, \alpha)$ unless $2\lambda - \alpha$ is a root of H.

b) if $\pi_{\mathbb{C}} = \pi_\lambda + \pi_\lambda^*$ then we also have $(\lambda, \lambda^*) = 0$ unless $\lambda + \lambda^*$ is a root of H.

c) $2\lambda - \alpha$ or $\lambda + \lambda^*$ is a root of H only for $G/H = S^n$, $P^n\mathbb{H}$, and $P^n\mathbb{C}$.

Conversely we have

Theorem 2: Let H be a compact Lie group

a) If π_λ is an irreducible orthogonal representation of H and $(\lambda, \lambda) = 2(\lambda, \alpha)$ whenever $(\lambda, \alpha) \neq 0$ and $2\lambda - \alpha \notin ad_H$ then π_λ is the isotropy representation of a symmetric space. If H is not semisimple we also need $(\lambda, \lambda^*) = 0$ and $\lambda + \lambda^* \notin ad_H$.

b) If π_λ is an irreducible symplectic representation with $2\lambda \notin ad_H$ then $(\lambda, \lambda) = \frac{3}{2}(\lambda, \alpha)$ whenever $(\lambda, \alpha) \neq 0$ iff $\pi_\lambda \otimes \overset{1}{o}$ is the isotropy

representation of a quaternionic symmetric space.

c) If H is semisimple and π_λ is an irreducible unitary representation with $\lambda + \lambda^* \notin ad_{H}$ then $(\lambda, \lambda) + (\lambda, \lambda^*) = 2(\lambda, \alpha)$ whenever $(\lambda, \alpha) \neq 0$ iff $\pi_\lambda \otimes \overset{1}{t} + \pi_\lambda{}^* \otimes \overset{-1}{t}$ is the isotropy representation of an irreducible hermitian symmetric space.

The space $SO(7)/G_2$, mentioned before as an exception, satisfies $(\lambda, \lambda) = 2(\lambda, \alpha)$, but not $2\lambda - \alpha \notin ad_H$, which explains why it does not come from a symmetric space.

Using the relationship $(\lambda, \lambda) = 2(\lambda, \alpha)$ for an irreducible symmetric space one can easily derive some properties of symmetric spaces which can be observed from a list of symmetric spaces but which are otherwise difficult to prove directly. If α is a simple root of a simple Lie group, we let τ_α be the fundamental representation corresponding to α.

Corollary 3: If G/H is irreducible symmetric, then

a) If H is not simple, then G/H is hermitian symmetric, quaternionic symmetric or a Grassmannian over $\mathbb{R}$ or $\mathbb{H}$. For each simple factor of H there exists only one simple root α wih $(\lambda, \alpha) \neq 0$. If G/H is quaternionic symmetric, then $\pi_\lambda = \tau_\alpha \otimes \overset{1}{o}$ for some α unless $\pi_\lambda = \overset{3}{o} \otimes \overset{1}{o}$. If G/H is hermitian symmetric, then $\pi_{\mathbb{C}} = \tau_\alpha \otimes \overset{1}{t} + \tau_\alpha \otimes \overset{-1}{t}$ for some α unless $\pi_{\mathbb{C}} = S^2\mu_n \otimes \overset{1}{t} + S^2\mu_n{}^* \otimes \overset{-1}{t}$.

b) If H is simple $\pi_\lambda = \tau_\alpha$ for some α unless $\pi_\lambda = ad_{SU(n)}$, $\overset{4}{o}$, $S^2\rho_n - id$, or $S^2\nu_{2n}$.

One can also use the relationship $(\lambda, \lambda) = 2(\lambda, \alpha)$ to give a quick classification of irreducible symmetric spaces according to their isotropy representation.

For our next theorem, let $\pi : H \to SO(n)$ be an orthogonal representation and let χ be the isotropy representation of $SO(n)/\pi(H)$. Then we have $\Lambda^2\pi = ad_H + \chi$ since $\Lambda^2\rho_n = ad_{SO(n)}$ and $ad_{SO(n)}|H = ad_H + \chi$, $\Lambda^2\rho_n|H = \Lambda^2\pi$. If π is an arbitrary representation of H and g a

biinvariant metric on h, we let $C_{\pi,g}$ be the Casimir operator

$$C_{\pi,g} = -\Sigma\, \pi(x_i)\, \pi(x_i)$$

where x_i is an orthonormal basis of $\mathfrak{h}$ w.r.t. g. Then we have

Theorem 4:

a) If G/H is an arbitrary symmetric space with isotropy representation $\pi : H \to SO(n)$ and $\Lambda^2 \pi = ad + \chi_V$, then $C_{\pi,-B_G} = \frac{1}{2}\,\mathrm{Id}$ and $C_{\chi,-B_G} = \mathrm{Id}$.

b) Conversely, if π is orthogonal and $\Lambda^2\pi = ad_H + \chi$ and if there exists a biinvariant metric g on $\mathfrak{h}$ such that $C_{\chi,g} = a \cdot Id$ for some a, then π is the isotropy representation of a symmetric space, unless π is equal to π_7 or $\pi_7 + \pi_7$, where π_7 is the seven dimensional representation of G_2.

In this theorem, the symmetric space, and hence π, does not have to be irreducible. Also, in general, χ will have many irreducible factors.

As an application of the last theorem we will show that if G/H is an irreducible symmetric space with H simple and with isotropy representation π, then $SO(n)/\pi(H)$ is strongly isotropy irreducible. Indeed, $\Lambda^2\pi = ad_H + \chi$ and we have to show that χ is irreducible. But in general, $\pi_{2\lambda-\alpha} \in \Lambda^2\pi_\lambda$ if $(\lambda, \alpha) \neq 0$ and unless $2\lambda - \alpha \in ad_H$ we have $\pi_{2\lambda-\alpha} \in \chi$. Any other summand in χ is a summand in $\Lambda^2\pi_\lambda$ and hence must have dominant weight $2\lambda - \alpha - \Sigma\, n_i\alpha_i$ where α_i are simple roots since there exists only one simple root α with $(\lambda, \alpha) \neq 0$. But the Casimir constant of this representation is smaller than the one for $\pi_{2\lambda-\alpha}$ as easily follows from $C_{\pi_\lambda,g} = g(\lambda, \lambda + 2\delta)\, Id$. But since $C_{\chi,-B_G} = Id$ this is impossible. Hence $\chi = \pi_{2\lambda-\alpha}$ and $SO(n)/\pi(H)$ is strongly isotropy irreducible.

This proof is a conceptual proof of the relationship between real symmetric spaces and strongly isotropy irreducible homogeneous spaces, at least in one direction. We have similar proofs for hermitian symmetric and quaternionic symmetric spaces. For a conceptual proof of the converse direction, one has to construct the symmetric space from a given strongly isotropy irreducible homogeneous space. We can do this so far for quotients of

$SO(n)$ and are presently working on the quotients of $SU(n)$ and $Sp(n)$. This problem is closely connected to giving a purely conceptual proof of Theorem 2. Similar remarks apply to the relationship between maximal subgroups of classical Lie groups and symmetric spaces. Proofs will be published elsewhere.

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