

CURVATURE HOMOGENEOUS MANIFOLDS IN DIMENSION 4

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ABSTRACT. We classify complete curvature homogeneous metrics on simply connected four dimensional manifolds which are invariant under a cohomogeneity one action.

Let M be a Riemannian manifold. M is called curvature homogeneous if, for any points $p, q \in M$, there exists a linear isometry $f : T_p M \rightarrow T_q M$ that preserves the curvature tensor, i.e. $f^* R_q = R_p$. In [Si]. I. Singer asked the question whether such manifolds are always homogeneous. The first complete counter examples were given by K.Sekigawa and H.Takagi in [Se, Ta], later generalized in [KTV]. The metric depends on several arbitrary functions of one variable and the curvature tensor is equal to the curvature tensor of the isometric product $\mathbb{H}^2 \times \mathbb{R}^{n-2}$, with a flat metric on \mathbb{R}^{n-2} . In [Br] it was shown that in dimension 3 there are complete examples which achieve any free group as their fundamental group, which can in addition depend on infinitely many arbitrary functions. There are many other local non-homogeneous examples which are curvature homogeneous, especially in dimension 3, see [BKV] and references therein. We point out though that the only known compact non-homogeneous examples are the Ferus-Karcher-Münzner isoperimetric hypersurfaces in \mathbb{S}^n , see [FKM].

It is natural to look for further examples among the class of cohomogeneity one manifolds, i.e. Riemannian manifolds on which a Lie G acts whose generic orbits are hypersurfaces. One such example was discovered by K.Tsukada, [Ts]. It is a complete metric on a two-dimensional vector bundle over $\mathbb{R}\mathbb{P}^2$, the normal bundle of the Veronese surfaces in $\mathbb{C}\mathbb{P}^2$. The Lie group $SO(3)$ acts on it by cohomogeneity one and the metric is given by

$$ds^2 = dt^2 + e^t d\theta_1^2 + e^{-t} d\theta_2^2 + (e^t + e^{-t}) d\theta_3^2$$

where t is the arc length parameter of a geodesic normal to all orbits, and $d\theta_i$ is the dual of the usual basis on the Lie algebra of $SO(3)$.

We will show that this example is indeed very special.

THEOREM. Let (M, G) be a four dimensional simply connected cohomogeneity one manifold with G a compact Lie group. Then any complete curvature homogeneous G -invariant metric is either isometric to a symmetric space, or to the Tsukada example.

Notice that this includes the case of warped product metrics of the form $dt^2 + g_t$ on \mathbb{R}^4 where g_t is an arbitrary left invariant metric on $SU(2)$, thus depending on 6 functions of one variable.

One can describe the proof as follows. The condition of being curvature homogeneous reduces to an ODE along the normal geodesic in terms of the metric and a G invariant connection. Since this system of ODE's depends on 9 functions, it is too complicated to solve for the metric directly. So we first discuss the case of diagonal metrics. In this case we will show that the components of the curvature tensor are constant, which implies that the metric is described by linear combinations of trigonometric, hyperbolic and linear functions. This gives rise to finitely many algebraic equations. If there exists a singular orbit, one can furthermore use the required smoothness conditions at the singular orbit, which makes it fairly straight forward to solve the

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system of equations. The result is that the metric is equivariantly isometric to one of the 8 cohomogeneity one actions on the 4-dimensional symmetric spaces or to the Tsukada example.

For a general metric we first prove that, using an equivariant diffeomorphism, the matrix of functions describing the metric can be partially diagonalized. This fact is true for any cohomogeneity one metric in all dimension and seems to be new. It may thus be of independent interest for other problems on cohomogeneity one manifolds. In our case this enables us to show that there exists an interval on the regular part (but not necessarily including the singular orbit) where the metric is diagonal. The resulting equations can be solved again, although the computations are now significantly more complicated since we cannot reduce the problem by using the smoothness conditions at a singular orbit. This will now also include the case where there are no singular orbits, where we show that there is no solutions at all.

The paper is organized as follows. In Section 1 we review the geometry of cohomogeneity one manifolds and discuss the condition of being curvature homogeneous. In Section 2 we show when the matrix of functions describing the metric can be partially diagonalized. In Section 3 we describe the known examples, in terms of the functions describing the metric, of the 4 dimensional cohomogeneity one metrics on symmetric spaces, as well as the example by Tsukada. In Section 4 we discuss the smoothness conditions and in Section 5 solve the case when the metric is diagonal. In Section 6 we discuss the general case.

1. PRELIMINARIES

Let M be a Riemannian manifold. Then M is curvature homogeneous if, for any $p, q \in M$, there exists a linear isometry $f : T_p M \rightarrow T_q M$ that preserves the curvature tensor, i.e. $f^* R_q = R_p$. This is equivalent (cfr. [TV]) to the existence of a metric linear connection D such that $DR = 0$. Parallel translation with respect to D can then be chosen as the isometry f above. Denote by ∇ the Levi-Civita connection of M , and let $A = \nabla - D$. Then A satisfies

$$(1.1) \quad \nabla R = A \cdot R.$$

To prove the existence of D one can simply argue as follows. Being curvature homogeneous implies that locally there exists an orthonormal frame field with respect to which the components of the curvature tensor are constant. Declaring this frame to be parallel gives a local solution to (1.1) and one can then use a partition of unity to define a global connection. Vice versa, the existence of a tensor A that satisfies (1.1) implies that M is curvature homogeneous.

If G acts by isometries such that $\dim(M/G) = 1$, a so called cohomogeneity one manifold, we can average the connection to make it G invariant, and still satisfy (1.1), see [V]. Thus it is sufficient to define the metric and the tensor A only along a geodesic. I.e., if $\gamma(t)$ is an arc length parameterized geodesic orthogonal to all hypersurface orbits of G , then M is curvature homogeneous if and only if there exists a skew-symmetric $(1, 1)$ -tensor $A(t)$, defined at the regular points of γ , such that

$$(1.2) \quad \begin{aligned} \frac{d}{dt} R(E_i, E_j, E_k, E_l) &= \sum_m a_i^m R(E_m, E_j, E_k, E_l) + a_j^m R(E_i, E_m, E_k, E_l) + \\ &+ a_k^m R(E_i, E_j, E_m, E_l) + a_l^m R(E_i, E_j, E_k, E_m) \end{aligned}$$

where $A(E_i) = \sum a_i^m E_m$ for a ∇ parallel orthonormal basis $E_i(t)$ of the tangent space at $\gamma(t)$. Thus if (1.2) is satisfied along the geodesic γ , the action of G guarantees that it is satisfied on all of M . Notice also that the tensor A needs to be defined only at the regular points, and we need to check (1.2) only at these regular points, since the metric is then curvature homogeneous on all of M by continuity. This avoids the technical issues of having to consider smoothness conditions at the singular point for A , in fact A may not even extend continuously to M . As we will see, it is also important for us to be able to change the normal geodesic γ by an equivariant diffeomorphism in order to simplify the computations.

Before we continue, let us recall the general structure of a cohomogeneity one manifold see, e.g., [AA, AB] for a general reference. Since we assume that M is simply connected, it follows that there are no exceptional orbits. The case where all orbits are regular is special and will be solved separately, see Section 6 case (b). So from now on we will assume that there exist some orbits which are singular. In that case a non-compact cohomogeneity one manifold is given by a homogeneous vector bundle and a compact one by the union of two homogeneous disc bundles. In our proof it will sufficient to solve the differential equation on a disc bundle near one singular orbit. The solutions will then determine whether the metrics extends to a complete metric on a vector bundle, or to a compact manifold. We will thus from now on also assume that we work simply on a single disc bundle. To describe the disc bundle, let H, K, G be compact Lie groups with inclusions $H \subset K \subset G$ such that $K/H = \mathbb{S}^\ell$ for some $\ell > 0$. The transitive action of K on \mathbb{S}^ℓ extends (up to conjugacy) to a unique linear action on $V = \mathbb{R}^{\ell+1}$, see e.g. [Be], Theorem 7.50. We can thus define the homogeneous vector bundle $M = G \times_K V = \{(g, v) \mid (g, v) \sim (gk^{-1}, kv) \text{ for any } k \in K\}$. G acts on M via $\bar{g}[g, v] = [\bar{g} \cdot g, v]$. A disc $\mathbb{D} \subset V$ can be viewed as the slice of the G action since, via the exponential map, it can be identified G equivariantly with a submanifold of M orthogonal to the singular orbit. Let $p_0 = [e, 0]$ be a point in the singular orbit $G \cdot p_0 = \{[g, 0] \mid g \in G\} \simeq G/K$ and $\gamma(t) = [e, te_0]$ a line in the slice V . Then the stabilizer group of G along $\gamma(t)$ is equal to K at p_0 and constant equal to H at $\gamma(t)$ for $t > 0$. Under the exponential map the image of γ is a geodesic in M orthogonal to all orbits. It is thus sufficient to describe G -invariant metrics on M only along $\gamma(t)$ since $G \cdot \gamma = M$. Conversely, given a cohomogeneity one manifold M , the slice theorem implies that the manifold in the neighborhood of a singular orbit has the above form after we choose a normal geodesic γ orthogonal to the singular orbit G/K with $\gamma(0) = p_0$.

We fix a bi-invariant metric Q on \mathfrak{g} , which defines a Q -orthogonal Ad_H invariant splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}$. The tangent space $T_{\gamma(t)}(G \cdot \gamma(t)) = \dot{\gamma}^\perp \subset T_{\gamma(t)}M$, is then identified with \mathfrak{n} for $t > 0$ via action fields: $X \in \mathfrak{n} \rightarrow X^*(\gamma(t))$. H acts on \mathfrak{n} via the adjoint representation and a G invariant metric on G/H is described by an Ad_H invariant inner product on \mathfrak{n} . For $t > 0$ the metric along γ is thus given by $g = dt^2 + h_t$ with h_t a one parameter family of Ad_H invariant inner products on the vector space \mathfrak{n} , depending smoothly on t . Conversely, given such a family of inner products h_t , we define the metric on the regular part of M by using the action of G . We describe the metric in terms of a one parameter family of self adjoint endomorphisms:

$$P_t: \mathfrak{n} \rightarrow \mathfrak{n}, \quad g(X^*(\gamma(t)), (Y^*(\gamma(t)))) = Q(P_t X, Y) \text{ for all } X, Y \in \mathfrak{n}.$$

Since Ad_H acts by isometries in g and Q , P_t commutes with Ad_H .

We choose an Ad_H invariant splitting

$$\mathfrak{n} = \mathfrak{n}_0 \oplus \mathfrak{n}_1 \oplus \dots \oplus \mathfrak{n}_r,$$

where Ad_H acts trivially on \mathfrak{n}_0 and irreducibly on \mathfrak{n}_i for $i > 0$. On $\mathfrak{n}_i, i > 0$ the inner product h_t is a multiple of Q , whereas on \mathfrak{n}_0 it is arbitrary. Furthermore, \mathfrak{n}_i and \mathfrak{n}_j are orthogonal if the representations of Ad_H are inequivalent. If they are equivalent, inner products are described by 1, 2 or 4 functions, depending on whether the equivalent representations are orthogonal, complex or quaternionic.

Next, we choose a basis X_i of \mathfrak{n} , adapted to the above decomposition, and thus the metrics h_t are described by a collection of smooth functions $g_{ij}(t) = g(X_i^*(\gamma(t)), X_j^*(\gamma(t))) = Q(P_t X_i, X_j)$, $t > 0$. In order to be able to extend this metric smoothly to the singular orbit, they must satisfy certain smoothness conditions at $t = 0$, which are discussed in [VZ].

Choosing an Ad_K invariant complement to $\mathfrak{k} \subset \mathfrak{g}$, we obtain the Q -orthogonal decompositions

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}, \quad \mathfrak{k} = \mathfrak{h} \oplus \mathfrak{p} \text{ and thus } \mathfrak{n} = \mathfrak{p} \oplus \mathfrak{m}.$$

where we can also assume that $\mathfrak{n}_i \subset \mathfrak{p}$ or $\mathfrak{n}_i \subset \mathfrak{m}$. Here \mathfrak{m} can be viewed as the tangent space to the singular orbit G/K at $p_0 = \gamma(0)$ and \mathfrak{p} as the tangent space of the sphere $K/H \subset V$. K acts via the isotropy action $\text{Ad}(K)|_{\mathfrak{m}}$ of G/K on \mathfrak{m} and via the slice representation on V . The (often

ineffective) linear action of K on V is determined by the fact that H is the stabilizer group at $\gamma(t)$, $t > 0$ and $K/H = \mathbb{S}^\ell$. We will also discuss in Section 6 the case where all orbits are regular.

It is important for us to note that the tubular neighborhood $G \times_K D$ is not only defined in a neighborhood of the singular orbit, but in fact for all t in the complete non-compact case, and until it reaches the second singular orbit in the compact case. Once we solve the condition for being curvature homogeneous near the singular orbit, the metric on M is well defined since the condition involving the tensor A is a regular ODE for $t > 0$. In practice, we will recognize the metric in a neighborhood of the singular orbit as a known example, and hence they must agree globally.

2. USE OF THE NORMALIZER

We will not try to solve the system (1.2) for the most general G -invariant metric on M since the conditions are too complicated. Instead we will use the degrees of freedom given by the action of the G -equivariant diffeomorphisms on M to show that, if a solution exists, then it is possible to find an isometric solution s.t. the expression of the metric is simpler.

2.1. Change of the metric on a homogeneous space. Let N be a Riemannian manifold on which a compact Lie group G acts transitively, almost effectively, and by isometries. If we fix a point $p_0 \in N$, and let H be the isotropy subgroup at p_0 , then N can be identified with G/H . We fix a bi-invariant metric Q on G and a Q -orthogonal decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{n}$ of \mathfrak{g} such that $T_{p_0}N \simeq \mathfrak{n}$ via action fields. The G -invariant metric g on N is identified with an endomorphism $P: \mathfrak{n} \rightarrow \mathfrak{n}$ via $g(X^*(p_0), Y^*(p_0)) = Q(P(X), Y)$ for all $X, Y \in \mathfrak{n}$.

The normalizer $L = N^G(H)/H$ acts, after fixing a base point, on the right, i.e. $R_n(gH) = gn^{-1}H$, or equivalently $R_n(gp_0) = gn^{-1}p_0$ for $n \in L$. L acts freely on N and transitively on the fixed point set N^H and thus $N^H \simeq L$. The Lie algebra of $N(H)$ is the centralizer of \mathfrak{h} in \mathfrak{g} and hence the Lie algebra of L can be identified with the subalgebra $\mathfrak{n}_0 = \{X \in \mathfrak{n} \mid \text{Ad}(h)X = X \text{ for all } h \in H\}$ in \mathfrak{g} . Notice that $\text{Ad}(g)(\mathfrak{n}_0) \subset \mathfrak{n}_0$ for all $g \in N(H)$ and hence L acts on \mathfrak{n}_0 via the adjoint representation. We can use this action to simplify the metric on \mathfrak{n}_0 .

First notice that for all $p \in N^H$ the Q -orthogonal complement \mathfrak{n} of \mathfrak{h} in \mathfrak{g} is independent of p (in fact this is true only for points in N^H). Hence we have the identification via action fields $T_qN \simeq \mathfrak{n}$ for all $q \in N^H$ for a *fixed* subspace \mathfrak{n} .

We now define a new metric $g' = R_n^*(g)$ for which we have:

$$\begin{aligned} Q(P'_t(X), Y) &= g'(X^*(p_0), Y^*(p_0)) = g(d(R_n)_{p_0}(X^*(p_0)), d(R_n)_{p_0}(Y^*(p_0))) \\ g(X^*(n^{-1}p_0), Y^*(n^{-1}p_0)) &= g(d(L_{n^{-1}})_{p_0}((\text{Ad}(n)X)^*(p_0)), d(L_{n^{-1}})_{p_0}((\text{Ad}(n)Y)^*(p_0))) \\ &= g((\text{Ad}(n)X)^*(p_0), (\text{Ad}(n)Y)^*(p_0)) = Q(P_t \text{Ad}(n)X, \text{Ad}(n)Y) = Q(\text{Ad}(n^{-1})P_t \text{Ad}(n)X, Y) \end{aligned}$$

since $X^*(gp) = d(L_g)_p((\text{Ad}(g)X)^*(p))$ and $d(R_n)_{p_0}(X^*(p_0)) = X^*(n^{-1}p_0)$.

Thus the change in the metric endomorphism is given by

$$(2.1) \quad P'_t = \text{Ad}(n^{-1})P_t \text{Ad}(n).$$

Notice that this can also be interpreted as changing the base point for g from p_0 to $n^{-1}p_0$.

In particular, if $\text{Ad}(L)|_{\mathfrak{n}_0} = \text{SO}(\mathfrak{n}_0)$ then we can assume that $P|_{\mathfrak{n}_0}$ is diagonal with respect to a Q orthonormal basis. Notice though that this is possible only if the identity component $L_0 \simeq \text{SO}(3)$ or $\text{SU}(2)$. Indeed, the action of L on \mathfrak{n}_0 is the adjoint action of L on its Lie algebra and if the image is the full orthogonal group, the maximal torus has dimension 1 since every vector of unit length can be conjugated into a fixed vector. This implies that the maximal abelian subalgebra of \mathfrak{l} is one dimensional.

2.2. Change of the metric on a Cohomogeneity one manifold. Let M be a cohomogeneity one Riemannian G -manifold as in Section 1, with $p_0 = \gamma(0) \in G/K$, principal isotropy group H , and Q -orthogonal decompositions $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n} = \mathfrak{h} + \mathfrak{p} + \mathfrak{m}$. This induces the identification $T_{\gamma(t)}G \cdot \gamma(t) \simeq \mathfrak{p} + \mathfrak{m}$ via action fields for all regular points.

We also have the subspaces $\mathfrak{p}_0 \subset \mathfrak{p}$ and $\mathfrak{m}_0 \subset \mathfrak{m}$ on which Ad_H acts as Id , and $L = N(H)/H$ acts on $\mathfrak{p}_0 \oplus \mathfrak{m}_0$ via its adjoint representation. As in the previous section we can try to change the metric on $\mathfrak{p}_0 \oplus \mathfrak{m}_0$ using the action of Ad_L . We want to understand if we can do this for all points along the normal geodesic smoothly. This will be easy on the regular part, but the smoothness at $t = 0$ is more delicate. Notice that for the metric endomorphisms P_t we have that $P_t(\mathfrak{p}_0 \oplus \mathfrak{m}_0) \subset \mathfrak{p}_0 \oplus \mathfrak{m}_0$ by Schur's Lemma since P_t commutes with Ad_H .

PROPOSITION 2.2. *Let (M, G) be a cohomogeneity one manifold with $L = N(H)/H$.*

- (a) *Assume that $L_0 = N(H)/H$ isomorphic to $SO(3)$ or $SU(2)$. If the action has a singular orbit at $t = 0$, then there exists an $\epsilon > 0$ such that any cohomogeneity one metric is G -equivariantly isometric to one where $P_t(\mathfrak{p}_0, \mathfrak{m}_0) = 0$ on $(0, \epsilon)$.*
- (b) *If $L_0 = N(H)/H$ isomorphic to $SO(3)$ or $SU(2)$ and there exists an interval $[a, b]$ with $a > 0$ on which the eigenvalues of $P_t|_{\mathfrak{n}_0}$ have constant multiplicity, then the metric is G -equivariantly isometric to one where $P_t|_{\mathfrak{n}_0}$ is diagonal for all $t \in (a, b)$.*
- (c) *Assume that $L_0 = N(H)/H$ is isomorphic to $SO(3)$, $SU(2)$ or $SO(2)$ respectively, and that there exists a three dimensional, respectively two dimensional Ad_H invariant subspace $\mathfrak{m}_1 \subset \mathfrak{n}_0^\perp \cap (\mathfrak{p} \oplus \mathfrak{m})$ which is also invariant under $\text{Ad}_{N(H)}$. If there exists an interval $[a, b]$ with $a > 0$ on which the eigenvalues of $P_t|_{\mathfrak{m}_1}$ have constant multiplicity, then the metric is G -equivariantly isometric to one where $P_t|_{\mathfrak{m}_1}$ is diagonal for all $t \in (a, b)$.*

Proof. (a) The assumption implies that $\dim(\mathfrak{p}_0 \oplus \mathfrak{m}_0) = 3$ and that Ad_L acts transitively on all Q -orthonormal basis in $\mathfrak{p}_0 \oplus \mathfrak{m}_0$. We will use this freedom to change the metric. For the transitive actions on spheres we have $\dim \mathfrak{p}_0 = 0, 1$ or 3 . If $\dim \mathfrak{p}_0 = 0$ or 3 there is nothing to prove. Thus we can assume from now on that $\dim \mathfrak{p}_0 = 1$. Notice that this happens precisely when K contains a normal subgroup isomorphic to $U(n)$, $SU(n)$ or $Sp(n) \cdot T^1$ acting linearly on the slice.

Along a normal geodesic the stabilizer group is constant at all regular points and hence, after fixing the choice of a principal isotropy group H , we can assume that for all G -invariant metrics a normal geodesics lies in M^H .

The normalizer $L = N(H)/H$ acts on the fixed point set M^H (on the left) and under this action M^H is also a cohomogeneity one manifold with $M/G = M^H/L$. On the regular part M_{reg}^H the stabilizer group is constant equal to H and hence we have the identification via action fields $T_q(G \cdot q) \simeq \mathfrak{p} \oplus \mathfrak{m}$ for all $q \in M_{reg}^H$ with respect to a *fixed* subspace $\mathfrak{p} \oplus \mathfrak{m} \subset \mathfrak{g}$. Under this identification $d(Lg|_{M_{reg}^H})_p = \text{Ad}(g)|_{\mathfrak{p} \oplus \mathfrak{m}}$ for all $p \in M_{reg}^H$ and $g \in N(H)$ since $d(Lg)_{p_0}(X^*(p_0)) = (\text{Ad}(g)X)^*(gp)$.

We also have an action of L on the right on M_{reg} (after fixing the geodesic γ) via $g\gamma(t) \rightarrow gn^{-1}\gamma(t)$ for $n \in L$. It acts freely on M_{reg} and transitively on each L -orbit in the fixed point set M_{reg}^H . In general this action will not extend to all of M though, unless n also normalizes K .

We will make use of the Weyl group element $\sigma \in K$ defined by $\sigma(p_0) = p_0$ and $\sigma(\gamma'(0)) = -\gamma'(0)$, and thus $\sigma(\gamma(t)) = \gamma(-t)$. Clearly, this defines σ uniquely mod H , and $\sigma^2 \in H$ as well as $\sigma \in N(H)$. According to the above, we also have $d(L\sigma|_{M_{reg}^H})_p = \text{Ad}(\sigma)|_{\mathfrak{p}_0 \oplus \mathfrak{m}_0}$, which will be useful for us when $p = \gamma(t)$.

Since $\text{Ad}(\sigma)$ normalizes Ad_H it preserves $\mathfrak{p}_0 \oplus \mathfrak{m}_0$ and since $\sigma^2 \in H$, it follows that $\text{Ad}(\sigma^2) = \text{Id}$ on $\mathfrak{p}_0 \oplus \mathfrak{m}_0$. Hence $\mathfrak{p}_0 \oplus \mathfrak{m}_0$ is the sum of two eigenspaces W_- and W_+ of $\text{Ad}(\sigma)$ corresponding to the eigenvalues ± 1 .

The geodesic $\gamma(t)$ and the metric P_t are defined and smooth on an interval around $t = 0$ and satisfy

$$P_{-t} = \text{Ad}(\sigma)P_t \text{Ad}(\sigma)^{-1}$$

since $\text{Ad}(\sigma) = d(L_\sigma)_{\gamma(t)} : T_{\gamma(t)}M \rightarrow T_{\gamma(-t)}M$ is an isometry and takes $\gamma(t)$ to $\gamma(-t)$. Notice that, although σ is only defined mod H , this is well defined since Ad_H commutes with P_t .

By assumption $\dim \mathfrak{p}_0 = 1$ and hence there exists a unique eigenvalue $\lambda_1(t)$ of P_t with $\lim_{t \rightarrow 0} \lambda_1(t) = 0$. Hence we can choose an $\epsilon > 0$ such that for $t \in (-\epsilon, \epsilon)$ the eigenvalue $\lambda_1(t)$ is distinct from all other eigenvalues of P_t . We restrict the remaining discussion to this interval only. Thus $\lambda_1(t)$ is a smooth function and there exists a smooth eigenvector $Y_1(t) \in \mathfrak{p}_0$ (see e.g. [L], Thm. 8, pg. 130) which we normalize to have unit length in Q . It follows that $\text{Ad}(\sigma)Y_1(t)$ is an eigenvector of P_{-t} with eigenvalue $\lambda(t)$ and hence there exists an $\epsilon = \pm 1$, independent of t , such that

$$Y_1(-t) = \epsilon \text{Ad}(\sigma)Y_1(t).$$

We want to show that this equation holds for a Q orthonormal basis of $\mathfrak{p}_0 \oplus \mathfrak{m}_0$, which will imply (2.4), and is crucial in defining the equivariant diffeomorphism ϕ later on.

Fix a Q -orthonormal basis $X_i \in \mathfrak{p}_0 \oplus \mathfrak{m}_0$ such that $\text{Ad}(\sigma)X_i = \epsilon_i X_i$ with $\epsilon_i = \pm 1$ and $X_1 = Y_1(0)$. We claim that we can extend X_i to three Q -orthonormal vector fields $Y_i(t)$ along γ with $Y_i(t) \in \mathfrak{p}_0 \oplus \mathfrak{m}_0$ and

$$(2.3) \quad Y_i(-t) = \epsilon_i \text{Ad}(\sigma)Y_i(t) \text{ and } Y_i(0) = X_i.$$

We already defined the vector field $Y_1(t)$ satisfying this equation. Next choose a vector field $Y_2(t) \in \mathfrak{p}_0 \oplus \mathfrak{m}_0$ orthogonal to $Y_1(t)$ with $Y_2(0) = X_2$ and define

$$Y_2(t) = \frac{1}{2}(Y_2(t) + \epsilon_2 \text{Ad}(\sigma)Y_2(-t))$$

Notice this implies that Y_2 satisfies (2.3) with $Y_2(0) = X_2$. We also have

$$\begin{aligned} 2Q(Y_1(t), Y_2(t)) &= 2Q(Y_1(t), Y_2(t) + \epsilon_2 \text{Ad}(\sigma)Y_2(-t)) = 2Q(Y_1(t), \epsilon_2 \text{Ad}(\sigma)Y_2(-t)) \\ &= 2Q(\text{Ad}(\sigma)Y_1(t), \epsilon_2 Y_2(-t)) = 2Q(\epsilon_1 Y_1(-t), \epsilon_2 Y_2(-t)) = 0 \end{aligned}$$

Similarly, we can find the vector field $Y_3(t) \in \mathfrak{p}_0 \oplus \mathfrak{m}_0$ satisfying (2.3) and orthogonal to $Y_1(t)$ and $Y_2(t)$. Finally, normalize $Y_i(t)$ to have unit length in Q .

Since Ad_L acts transitively on the set of Q -orthonormal basis in $\mathfrak{p}_0 \oplus \mathfrak{m}_0$, there exists an element $n_t \in L$ such that

$$\text{Ad}(n_t)X_i = Y_i(t) \text{ for } t \in (-\epsilon, \epsilon).$$

Furthermore, n_t is uniquely determined since the center of L is finite and $n_0 = e$. Thus n_t depends smoothly on t since $\text{Ad}(n_t)_{\gamma(t)}$ is an isomorphism for all $t \in (-\epsilon, \epsilon)$. Finally, we choose $\epsilon' > 0$ such that $n_t = e$ for $t > \epsilon + \epsilon'$. Here we point out that X_i should be interpreted as action fields along γ , whereas $Y_i(t)$ are simply vector fields along γ .

We next observe that the curve n_t has the crucial property

$$(2.4) \quad n_{-t} = \sigma \cdot n_t \cdot \sigma^{-1}$$

modulo elements of H since (2.3) implies that

$$\text{Ad}(n_{-t}) \text{Ad}(\sigma)X_i = \epsilon_i \text{Ad}(n_{-t})X_i = \epsilon_i Y_i(-t) = \text{Ad}(\sigma)Y_i(t) = \text{Ad}(\sigma) \text{Ad}(n_t)X_i$$

for all i .

Using the curve n_t in L , we define a map $\phi : M_{reg} \rightarrow M_{reg}$ by using the right action of L on each orbit:

$$\phi(g\gamma(t)) = gn_t^{-1}\gamma(t), \quad t \in (-\epsilon, \epsilon), \quad t \neq 0.$$

This is clearly well defined for a fixed $t > 0$ since Ad_L acts freely on each regular orbit. Furthermore, each G orbit intersects the geodesic in precisely two points, namely $\gamma(t)$ and $\sigma\gamma(t) = \gamma(-t)$. To see that ϕ is well defined on M_{reg} we use (2.4):

$$\phi(\sigma\gamma(t)) = \sigma n_t^{-1} \gamma(t), \text{ and } \phi(\gamma(-t)) = n_{-t}^{-1} \gamma(-t) = n_{-t}^{-1} \sigma\gamma(t) = \sigma n_t^{-1} \gamma(t).$$

Finally, ϕ can be extended to the singular orbit since $n_0 = e$. Altogether this implies that $\phi: M \rightarrow M$ is well defined, continuous, and G -equivariant. It is smooth on M_{reg} since $M_{reg} = I \times G/H$ for some open interval I and $\phi(t, gH) = (t, gn_t^{-1}H)$ which is smooth since n_t is. To see that it is also smooth at the singular orbit we can argue as follows. Let $U(0) \subset \mathfrak{m}$ be a neighborhood of 0 with $\bar{U} = \exp_G(U(0))$ such that $\bar{U} \rightarrow G/K$, $g \rightarrow gK$ is a diffeomorphism onto its image. Then, if $\bar{V} \subset V$ is also a small neighborhood of 0 in the slice, the map $(g, v) \rightarrow gv$ is a diffeomorphism of $\bar{U} \times \bar{V}$ onto a small neighborhood of $\gamma(0)$ in M . In these coordinates, $(e, te_0) = \gamma(t)$ and ϕ becomes $g \cdot \gamma(t) = g(e, te_0) = (g, te_0) \rightarrow (gn_t^{-1}, te_0)$ and is hence smooth.

We now change the metric g to the new metric $\bar{g} = \phi^*(g)$ for which the curve $\bar{\gamma}(t) = n_t^{-1}\gamma(t)$ is a geodesic normal to all orbits. The metric endomorphism P_t with respect to the geodesic $\bar{\gamma}$, changes, according to (2.5), into

$$(2.5) \quad \bar{P}_t = \text{Ad}(n^{-1}) P_t \text{Ad}(n)$$

and since $\text{Ad}(n_t)X_i = Y_i(t)$ we get

$$\bar{P}_t(X_1) = \text{Ad}(n_t^{-1})P_t \text{Ad}(n_t)X_1 = \text{Ad}(n_t^{-1})P_t Y_1(t) = \text{Ad}(n_t^{-1})\lambda_1(t)Y_1(t) = \lambda_1(t)X_1.$$

X_2 and X_3 belong to a different eigenspace of \bar{P}_t . Thus the metric \bar{P}_t is diagonal.

(b) and (c) The proof in these cases works similarly, in fact is simpler since we are staying away from the singular orbit. Choose $\epsilon > 0$ such the eigenvalues on $(a - \epsilon, b + \epsilon)$ have constant multiplicity. This implies that the eigenvectors are smooth and we choose, for $t \in (a - \epsilon, b + \epsilon)$, an orthonormal basis of eigenvectors of $P_t|_{\mathfrak{p}_0 \oplus \mathfrak{m}_0}$, respectively $P_t|_{\mathfrak{m}_1}$. Let $X_i = Y_i(0)$ considered as action fields along γ . Choose a smooth curve $n_t \in N(H)/H$ such that $\text{Ad}(n_t)X_i = Y_i(t)$ for $t \in (a - \epsilon/2, b + \epsilon/2)$ and extend the curve such that $n_t = \text{Id}$ outside $(a - \epsilon, b + \epsilon)$. This smooth curve n_t defines a G -equivariant diffeomorphism $\phi(g\gamma(t)) = gn_t^{-1}\gamma(t)$ and the rest of the proof is as before since $n_t = \text{Id}$ on an interval around $t = 0$. \square

Notice though that in general $\text{Ad}_{N(H)}$ only takes Ad_H invariant subspaces into other Ad_H invariant subspaces but does not necessarily preserve them unless they are not equivalent to any other Ad_H invariant subspace. Notice also that in part (b) and (c) in fact no singular orbit is required.

Remark 2.6. The proof also shows that one can change the normal geodesic via an equivariant diffeomorphism into another curve transverse to all orbits. Indeed, let $\gamma_i(t)$, $i = 1, 2$ be two smooth curves in M^H transverse to all orbits satisfying $\gamma_i(-t) = \sigma_i\gamma_i(t)$, where σ_i is the Weyl group element at $\gamma_i(0)$. Then there exists a function f with $f(t) = f(-t)$ and a smooth curve $n_t \in N(H)/H$ such that $\gamma_2(t) = n_t^{-1}\gamma_1(f(t))$ at the regular points. The map $\phi(g\gamma_1(t)) = gn_t^{-1}\gamma_2(f(t))$ is an equivariant diffeomorphism taking γ_1 to γ_2 . The condition (2.4) becomes $\sigma_2 n_{-t} = n_t \sigma_1$ which follows from $\gamma_i(-t) = \sigma\gamma_i(t)$. The well definedness and smoothness of ϕ then follows as above. Of course if there are no non-regular orbits there are no conditions. In particular, in order to describe the set of all G -invariant metrics one can fix a curve and its parametrization, such that it is a geodesic normal to the orbits for all G -invariant metrics, up to isometry.

3. FOUR DIMENSIONAL COHOMOGENEITY ONE MANIFOLDS

In this section we describe all simply connected 4-dimensional cohomogeneity one manifolds (M^4, G) with G compact, and such that not all orbits are regular. Thus there exists a singular isotropy group, which we denote by K . The principal orbit G/H is 3 dimensional and hence $G = SO(4), U(2), SU(2), SO(3)$ or T^3 . If $G = T^3$ we can assume the principal isotropy group is trivial, since otherwise G does not act effectively on G/H and hence on M . Thus $K = S^1$, hence each disk bundle is homotopy equivalent to $T^3/S^1 = T^2$, where as the principal orbit is T^3 . In the compact case this contradicts van Kampen. In the first four cases, G contains a normal subgroup $SU(2)$ or $SO(3)$ which acts transitively on G/H . Thus it is also cohomogeneity one under an almost effective action of $SU(2)$ and we will from now on assume that $G = SU(2)$. Hence the principal orbit is $SU(2)/H$ with H a finite group. Such finite groups are isomorphic to \mathbb{Z}_k , a binary dihedral group D_k^* (of order $4k$), or the binary groups T^*, O^*, I^* . For the three latter groups the isotropy representation of Ad_H on \mathfrak{n} is irreducible. This means that the only possibility for K is G itself, contradicting the assumption that K/H is a sphere. If $H = \mathbb{Z}_k$, then $K = SO(2)$ or $\text{Pin}(2)$, and if $H = D_k^*$, then $K = \text{Pin}(2)$ are the only possibilities, apart from the special case where $G = K$ and $H = \{e\}$, i.e., where the action has a fixed point.

For the smoothness conditions of the metric near a codimension two singular orbit, i.e. $K_0 = S^1$, one needs to consider two integers depending on the action of K_0 on \mathfrak{m} and on the slice, see [VZ]. In our case they are both two dimensional and K_0 acts with speed 2 on \mathfrak{m} in all cases, and with speed $a = |H \cap K_0|$ on the slice. The integer a depends on the group diagram and will be crucial in our discussion.

We now describe the isotropy representation, the metrics, and the integer a . For convenience we identify $SU(2)$ with $Sp(1)$.

a) $K = SO(2) = \{e^{i\theta}\}$ and $H = \mathbb{Z}_k$. The group H is generated by $\zeta = e^{\frac{2\pi i}{k}}$, and hence $\mathfrak{n} = \mathfrak{n}_0 \oplus \mathfrak{n}_1 \simeq \mathbb{R} \oplus \mathbb{C}$ and Ad_H acts on \mathfrak{n} as $\text{Ad}(\zeta)(r, z) = (r, \zeta^2 z)$. If $k = 2$, H is effectively trivial and any metric on \mathfrak{n} is allowed. If $k = 4$, the action is $(r, z) \rightarrow (r, -z)$ and hence \mathfrak{n}_0 is orthogonal to \mathfrak{n}_1 and on the two dimensional module \mathfrak{n}_1 , the metric can be arbitrary. In all other cases the metric is diagonal and depends on 2 functions. K acts on $\mathfrak{m} \simeq \mathbb{C}$ as $v \rightarrow A^2 v$ since G/K is \mathbb{Z}_2 ineffective. Its action on the slice $V \simeq \mathbb{C}$ is given by $v \rightarrow A^k v$ since K/H is \mathbb{Z}_k ineffective. Hence $a = k$.

(b) $K = \text{Pin}(2) = \{e^{i\theta}\} \cup j \cdot \{e^{i\theta}\}$ and $H = \mathbb{Z}_k$. In order for $K/H = S^1$, H must contain an element in the second component of K , which we can assume is $j \in Sp(1)$. Hence necessarily $k = 4$, and H is generated by j . The action of H on $\mathfrak{n} = \mathfrak{n}_0 \oplus \mathfrak{n}_1 \simeq \mathbb{R} \oplus \mathbb{C}$ is $(r, z) \rightarrow (r, -z)$ and hence the metric on \mathfrak{n}_1 is arbitrary. Here $a = 2$.

(c) $K = \text{Pin}(2) = \{e^{i\theta}\} \cup j \cdot \{e^{i\theta}\}$ and $H = D_k^*$, $k > 1$ generated by $\zeta = e^{\frac{2\pi i}{2k}}$ and j . If $k = 2$, H is the quaternion group. The action of Ad_H on $\mathfrak{n} = \mathfrak{n}_0 \oplus \mathfrak{n}_1 \simeq \mathbb{R} \oplus \mathbb{C}$ is given by $\text{Ad}(\zeta)(r, z) = (r, \zeta^2 z)$ and $j \cdot (r, z) = (r, b + ci) \rightarrow (-r, b - ci)$. The metric is diagonal and depends on 3 functions. If $k > 2$, it depends on only 2 functions. Here $a = 2k$.

(d) The last possibility is a codimension 4 singular orbit, i.e., $G = K = SU(2)$ and hence $H = \{e\}$.

A compact manifold is (since we assume not all orbits are regular) the union of two such disk bundles. But in our discussion it will be sufficient to solve the problem near one singular orbit.

We choose the following basis for the Lie algebra of $SU(2)$:

$$X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad X_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

with Lie brackets:

$$[X_i, X_j] = 2X_k, \text{ where } i, j, k \text{ is a cyclic permutation of } 1, 2, 3.$$

We fix a bi-invariant metric Q on \mathfrak{g} such that X_i is orthonormal. The metric will be determined by the inner products of these basis elements.

We need to compare our solutions with the known solutions on simply connected homogeneous 4 dimensional manifolds with $G = SU(2)$, considered as a cohomogeneity one manifold. It turns out that these are all symmetric spaces. For the compact examples the metric is defined on $0 \leq t \leq t_0$ with stabilizer groups K_- and K_+ at $t = 0$ and $t = t_0$. An upper index ' indicates a different embedding. See [Zi] for details. Notice though that since we choose $G = SU(2)$ instead of $G = SO(3)$ as in [Zi], the functions must be multiplied by 2. Since all metrics are diagonal, we list the length of the basis X_i , i.e. $f_i = v_i^2$.

Ineffective kernel $a = 4$

Example 1 $M = \mathbb{S}^4$ with $G = SU(2)$, $K_- = \text{Pin}(2)$, $K_+ = \text{Pin}(2)'$ and $H = D_2^*$, the quaternion group, and metric

$$v_1 = 4 \sin(t), \quad v_2 = 2\sqrt{3} \cos(t) - 2 \sin(t), \quad v_3 = 2\sqrt{3} \cos(t) + 2 \sin(t) \quad \text{with } 0 \leq t \leq \pi/3$$

Example 2 $M = \mathbb{C}\mathbb{P}^2$ with $G = SU(2)$, $K_- = SO(2)'$, $K_+ = \text{Pin}(2)$ and $H = \mathbb{Z}_4$ and metric

$$v_1 = 2 \sin(2t), \quad v_2 = \sqrt{2}(\cos(t) - \sin(t)), \quad v_3 = \sqrt{2}(\cos(t) + \sin(t)) \quad \text{with } 0 \leq t \leq \pi/4$$

Example 3 Tsukada Example with $G = SU(2)$, $K_- = \text{Pin}(2)$ and $H = D_2^*$ and metric

$$v_1 = 2(e^t - e^{-t}), \quad v_2 = 2e^t, \quad v_3 = 2e^{-t}$$

Ineffective kernel $a = 2$

Example 4 $M = \mathbb{C}\mathbb{P}^2$ with $G = SU(2)$, $K_- = \text{Pin}(2)$, $K_+ = SO(2)'$ and $H = \mathbb{Z}_4$ and metric

$$v_1 = 2 \sin(t), \quad v_2 = 2 \cos(2t), \quad v_3 = 2 \cos(t) \quad \text{with } 0 \leq t \leq \pi/4$$

Example 5 $M = \mathbb{S}^2 \times \mathbb{S}^2$ with $G = SU(2)$, $K_- = SO(2)$, $K_+ = SO(2)$ and $H = \mathbb{Z}_2$ and metric

$$v_1 = 2 \sin(t), \quad v_2 = 2, \quad v_3 = 2 \cos(t) \quad \text{with } 0 \leq t \leq \pi/2$$

Ineffective kernel $a = 1$ (actions with fixed points)

Example 6 $M = \mathbb{C}\mathbb{P}^2$ with $G = SU(2)$, $K_- = SO(2)$, $K_+ = SU(2)$ and $H = \{e\}$ and metric

$$v_1 = \frac{1}{2} \sin(2t), \quad v_2 = v_3 = \cos(t), \quad \text{with } 0 \leq t \leq \pi/2$$

Example 7 $M = \mathbb{C}\mathbb{P}^2$ with $G = SU(2)$, $K_- = SU(2)$, $K_+ = SO(2)$ and $H = \{e\}$ and metric

$$v_1 = v_2 = \sin(t), \quad v_3 = \frac{1}{2} \sin(2t) \quad \text{with } 0 \leq t \leq \pi/2$$

Example 8 $M = \mathbb{C}\mathbb{H}^2$ with $G = K = SU(2)$ and $H = \{e\}$ and metric

$$v_1 = v_2 = \sinh(t), \quad v_3 = \frac{1}{2} \sinh(2t) \quad \text{with } 0 \leq t < \infty$$

Example 9 $M = \mathbb{S}^4$ with $G = SU(2)$, $K_- = K_+ = SU(2)$ and $H = \{e\}$ and metric

$$v_1 = v_2 = v_3 = \sin(t) \quad \text{with } 0 \leq t \leq \pi$$

Example 10 $M = \mathbb{H}^4$ with $G = SU(2)$, $K_- = K_+ = SU(2)$ and $H = \{e\}$ and metric

$$v_1 = v_2 = v_3 = \sinh(t) \quad \text{with } 0 \leq t \leq \infty$$

Example 11 $M = \mathbb{R}^4$ with $G = K = SU(2)$ and $H = \{e\}$.

$$v_1 = v_2 = v_3 = t \quad \text{with } 0 \leq t < \infty$$

Example 2 and 4, as well as Example 6 and 7, are the same action and metric, but the two singular orbits are interchanged. The functions look different though, which will be useful when solving the equations in the next few Sections.

In [Ts] Tsukada uses $G = SO(3)$ instead of $SU(2)$ which means the functions in Example 3 need to be divided by 2 when comparing. The metric is clearly complete, and Tsukada proved that it is not homogeneous by showing that $|\nabla \text{Ric}|$ is not constant. The underlying non-compact manifold can be regarded as one of the homogeneous disc bundles in the Example 1 on \mathbb{S}^4 . All other examples are isometric to symmetric spaces.

This list of Examples is also a description of all cohomogeneity one actions on simply connected 4-manifolds with $G = SU(2)$ and not all orbits regular, as long as we add the action of $SU(2)$ on $\mathbb{C}\mathbb{P}^2 \# -\mathbb{C}\mathbb{P}^2$. In this case $K_- = K_+ = SO(2)$, $H = \{e\}$ and $a = 1$. But, according to our result, it does not admit a curvature homogeneous metric.

4. THE FORM OF THE METRIC

In this section we discuss the metric and the smoothness conditions. The metric is completely describe by the restriction to the space tangent to the regular orbits along the normal geodesic, and has the form

$$f_i(t) = g_t(X_i^*, X_i^*)_{\gamma(t)}, \quad d_{ij}(t) = g_t(X_i^*, X_j^*)_{\gamma(t)}.$$

for $i, j = 1, 2, 3$ and $i \neq j$. We identify g_t with the matrix

$$P_t = \begin{pmatrix} f_1 & d_{12} & d_{13} \\ d_{12} & f_2 & d_{23} \\ d_{13} & d_{23} & f_3 \end{pmatrix}.$$

If the singular orbit has codimension two, we have:

$$\mathfrak{g} = \mathfrak{p}_0 + \mathfrak{m}$$

where $\dim \mathfrak{p}_0 = 1$, $\dim \mathfrak{m} = 2$ and possibly $\mathfrak{m} = \mathfrak{m}_0$. Let

For the smoothness conditions the integers $d = 2$ and a (see the notaion in [VZ]) enter. According to [VZ] the metric P_t is smooth at $t = 0$ if and only if:

$$(4.1) \quad \begin{aligned} f_1 &= a^2 t^2 + t^4 \phi_1(t^2), & f_2 + f_3 &= \phi_2(t^2), & f_2 - f_3 &= t^{\frac{4}{a}} \phi_3(t^2) \\ d_{12} &= t^{2+\frac{2}{a}} \phi_4(t^2), & d_{13} &= t^{2+\frac{2}{a}} \phi_5(t^2) & d_{23} &= t^{\frac{4}{a}} \phi_6(t^2) \end{aligned}$$

for some smooth functions ϕ_i .

It follows that the metric is diagonal and $f_2 = f_3$ unless $a = 1, 2, 4$ and $d_{12} = d_{13} = 0$ unless $a = 1, 2$. For $a \leq 4$ we have the following possibilities for K and H according to Section 3:

- (1) If $a = 1$, then $K = SO(2)$ and $H = \{e\}$.
- (2) If $a = 2$, then $K = \text{Pin}(2)$ and $H = \mathbb{Z}_4$, or $K = SO(2)$ and $H = \mathbb{Z}_2$.
- (3) If $a = 4$, then $K = \text{Pin}(2)$ and $H = D_2^*$, or $K = SO(2)$ and $H = \mathbb{Z}_4$.

If $H = \{e\}$ or $H = \mathbb{Z}_2$ we have $L = N_G(H)/H = SU(2)$ and H acts trivially on $\mathfrak{m} = \mathfrak{m}_0$. In this case Proposition 2.2 implies that we can assume, up to a equivariant diffeomorphism, $d_{12} = d_{13} = 0$ in a neighborhood of the singular orbit. If $a = 2$ and $H = \mathbb{Z}_4$ the Ad_H invariance

already implies that $d_{12} = d_{13} = 0$. If $a = 4$, any invariant metric is diagonal. Thus we can always assume that $d_{12} = d_{13} = 0$.

If the singular orbit has codimension 4, smoothness according to [VZ] means that

$$f_i = t^2 + t^4 \phi_i(t^2), \quad d_{ij} = t^4 \phi_{ij}(t^2)$$

for some smooth functions ϕ_i, ϕ_{ij} .

The last case, where the action has no singular orbits, no smoothness conditions are required, apart from the functions f_i, d_{ij} being smooth for all t .

We finally remark that if the manifold is non-compact, the metric is complete if it is defined for all $t \in \mathbb{R}$.

5. DIAGONAL METRICS

In this Section we consider smooth diagonal metrics, i.e., metrics where $d_{ij} = 0$, on simply connected cohomogeneity one manifolds with one singular orbit. The case where the action has a fixed point is easy and left to the end of the section and the case where all orbits are regular will be dealt with separately in Section 6. So we assume that $G = SU(2)$ and $K_0 = SO(2)$. Here it will be simpler to express the functions as length $f_i = v_i^2$. Then the vector fields $E_i = \frac{1}{v_i} X_i$, for $i = 1, 2, 3$, and $E_4 = \gamma'(t)$ form an orthonormal basis of $T_{\gamma(t)}M$ for all $t \geq 0$. In this case the only non-zero components of the curvature tensor are (see [GVZ]):

$$(5.1) \quad \begin{aligned} R(E_i, E_j, E_i, E_j) &= \frac{2v_k^2(v_i^2 + v_j^2) - 3v_k^4 + (v_i^2 - v_j^2)^2}{v_i^2 v_j^2 v_k^2} - \frac{v_i' v_j'}{v_i v_j} \\ R(E_i, E_j, E_k, E_4) &= \sigma \left(-2 \frac{v_k'}{v_i v_j} + \frac{v_i' v_i^2 + v_k^2 - v_j^2}{v_i v_i v_j v_k} + \frac{v_j' v_j^2 + v_k^2 - v_i^2}{v_j v_i v_j v_k} \right) \\ R(E_i, E_4, E_i, E_4) &= -\frac{v_i''}{v_i} \end{aligned}$$

where σ is the sign of the permutation (i, j, k) .

In order to apply (1.2), we make the following general observation:

LEMMA 5.2. *Let M be a Riemannian cohomogeneity one manifold, and X_i^* Killing vector fields which form a basis of $(\gamma')^\perp$ along a geodesic γ normal to all orbits. If X_i^* are orthogonal to each other along γ , then the normalized vector fields $\bar{X}_i = X_i^*/|X_i^*|$ are parallel along γ .*

Proof. Since $\langle X_i^*, X_j^* \rangle = 0$ for $i \neq j$, it follows, using $\langle [X_i^*, X_j^*], \gamma' \rangle = 0$, that

$$\begin{aligned} 0 &= \langle X_i^*, X_j^* \rangle' = \langle \nabla_{\gamma'} X_i^*, X_j^* \rangle + \langle X_i^*, \nabla_{\gamma'} X_j^* \rangle = \langle \nabla_{X_j^*} X_i^*, \gamma' \rangle + \langle \nabla_{X_i^*} X_j^*, \gamma' \rangle \\ &= 2 \langle \nabla_{X_j^*} X_i^*, \gamma' \rangle = -2 \langle \nabla_{\gamma'} X_i^*, X_j^* \rangle = -2 |X_i^*| \langle \nabla_{\gamma'} \bar{X}_i, X_j^* \rangle. \end{aligned}$$

Furthermore, $\langle \nabla_{\gamma'} \bar{X}_i, \bar{X}_i \rangle = \frac{1}{2} \langle \bar{X}_i, \bar{X}_i \rangle' = 0$, and $\langle \nabla_{\gamma'} \bar{X}_i, \gamma' \rangle = \frac{1}{|X_i^*|} \langle \nabla_{\gamma'} X_i^*, \gamma' \rangle = 0$. Altogether, we have $\nabla_{\gamma'} \bar{X}_i = 0$. \square

Since action fields are Killing vector fields along a geodesic, it follows that E_i are parallel along γ .

From (5.1) it follows that if $R(E_i, E_j, E_k, E_l) \neq 0$, then $R(E_m, E_j, E_k, E_l) \neq 0$ if and only if $m = i$. Together with the skew symmetry of A , it follows that the right hand side in (1.2) is 0. Hence, if M is curvature homogeneous, all components of the curvature tensor are constant (for a diagonal metric). We will use this property and take a power series at the singular orbit to classify the metrics.

We start with the case where the singular orbit has codimension 2.

At the singular orbit $v_1(t)$ satisfies $v_1(0) = 0$. The equation $R(E_1, E_4, E_1, E_4) = -\frac{v_1''}{v_1} = c$ can be integrated explicitly, and we can normalize the metric so that $c = 1, -1$ or 0 . Thus v_1 must be one of $a \sin(t)$, $a \sinh(t)$ or at for some constant a , determined by the ineffective kernel of the action. We can also solve the equations $R(E_1, E_i, E_1, E_i) = -\frac{v_i''}{v_i} = c_i$, $i = 2, 3$ for some constants c_i . Notice that when v_1 is a trigonometric function the manifold is compact. Since v_1 vanishes at $k\pi$, $k \in \mathbb{Z}$, the stabilizer group is equal to $SO(2)$ at these points and hence v_2 and v_3 must agree at $k\pi$ since the metric is invariant under $SO(2)$ at these points. Thus v_2 and v_3 are trigonometric functions as well. Similarly, if v_1 is hyperbolic or linear, M is non-compact and hence v_2 and v_3 are both hyperbolic or linear as well.

Hence the metric is determined by a few constants. We can plug the functions v_i in the expressions for the components of the curvature tensor, that have to be constant. By computing the first non-zero derivatives of the curvatures at $t = 0$ we derive some algebraic conditions on these constants which can be solved explicitly.

It is convenient to first work first with the special case of $v_2(t) = v_3(t)$.

5.1. $v_2(t) = v_3(t)$. Notice that smoothness implies that v_2 must be an even function. We also have that $R(E_2, E_4, E_2, E_4) = R(E_3, E_4, E_3, E_4) = c$. If M is compact, the smoothness conditions imply that

$$v_1 = a \sin(t), \quad v_2 = b \cos(ct) \quad v_3 = b \cos(ct)$$

with $a, b, c > 0$. Computing the first non-zero derivatives of $R(E_1, E_2, E_1, E_2)$ and $R(E_2, E_3, E_2, E_3)$ at $t = 0$ gives us two equations:

$$b^4 c^4 - 4b^2 c^2 + 3a^2 = 0, \quad b^4 c^4 - b^4 c^2 + 3a^2 = 0$$

which implies $b^c c^2 (b^2 - 4) = 0$. Thus $b = 2$, hence $16c^4 - 16c^2 + 3a^2 = 0$, which implies that $a = 1$, and $c = \frac{1}{2}$. Thus the only solution is

$$v_1 = \sin(t), \quad v_2 = 2 \cos\left(\frac{1}{2}t\right) \quad v_3 = 2 \cos\left(\frac{1}{2}t\right)$$

which is, up to reparametrization of the normal geodesic, Example 6 on $\mathbb{C}\mathbb{P}^2$.

If M is non-compact, we have one possibility:

$$v_1 = a \sinh(t), \quad v_2 = b \cosh(ct) \quad v_3 = b \cosh(ct)$$

with $a, b > 0$, $c \geq 0$. This gives rise to the equations: $b^4 c^4 + 4b^2 c^2 + 3a^2 = 0$, $b^4 c^4 - b^4 c^2 + 3a^2 = 0$, thus $b^c c^2 (b^2 + 4) = 0$, which forces $a = 0$ and we have no solutions.

The last possibility is that

$$v_1 = at, \quad v_2 = b \cosh(ct) \quad v_3 = b \cosh(ct)$$

with $a, b > 0$, $c \geq 0$. This gives rise to the equations: $b^4 c^4 + 4b^2 c^2 + 3a^2 = 0$, $b^4 c^4 + 3a^2 = 0$ which again has only 0 solutions.

From now on we may assume that $v_2 \neq v_3$. The smoothness conditions at the singular orbit tells us that

$$v_2^2 + v_3^2 = \phi_1(t^2), \quad v_2^2 - v_3^2 = t^{\frac{4}{a}} \phi_2(t^2)$$

for some smooth functions ϕ_1, ϕ_2 . If $\frac{4}{a}$ is not an integer this implies $v_2 = v_3$, hence we may assume that $a = 1, 2, 4$. But for $a = 1$ we have that v_2 and v_3 must agree in 0 up to order 3. Since both functions satisfy the ODE $R(E_i, E_4, E_i, E_4) = c_i$, it again follows that $v_2 = v_3$. Hence we are just left with the cases $a = 2$ and $a = 4$. We consider the cases $c = 1, 0, -1$ separately. Notice also that the smoothness conditions imply that if v_2 is hyperbolic and v_3 linear, then v_3 must be constant.

5.2. $c = 1$. Now $v_1 = a \sin(t)$ with $a = 2$ or $a = 4$.

5.2.1. $a=2$. The smoothness conditions imply that v_2 and v_3 are even with $v_2(0) = v_3(0) \neq 0$ and hence

$$v_1 = 2 \sin(t), \quad v_2 = b \cos(c_1 t), \quad v_3 = b \cos(c_2 t)$$

for some constants $b > 0, c_i \geq 0$. From the first non-zero derivatives of $R(E_1, E_2, E_1, E_2)$, $R(E_1, E_3, E_1, E_3)$, $R(E_2, E_3, E_2, E_3)$ we get the equations:

$$A = b^4 c_1^4 - 6b^4 c_1^2 c_2^2 + b^4 c_2^4 + 8b^2 c_1^2 + 8b^2 c_2^2 - 48 = 0$$

$$B = b^4 c_1^4 - 6b^4 c_1^2 c_2^2 + b^4 c_2^4 - 4b^4 c_1^2 + 8b^4 c_2^2 + 24b^2 c_1^2 - 24b^2 c_2^2 - 48 = 0$$

$$C = b^4 c_1^4 - 6b^4 c_1^2 c_2^2 + b^4 c_2^4 + 8b^4 c_1^2 - 4b^4 c_2^2 - 24b^2 c_1^2 + 24b^2 c_2^2 - 48 = 0$$

Hence $\frac{4}{3}B - \frac{1}{3}C - A = 8b^2(c_1^2 - \frac{3}{2}c_2^2)(4 - b^2) = 0$, $B - C = 12b^2(c_1^2 - c_2^2)(4 - b^2) = 0$, and thus $b = 2$ (in which case the 3 equations are identical). Using in addition the derivatives of the remaining components of the curvature tensor one obtains:

$$(c_1^2 - c_2^2)^2 - 2(c_1^2 + c_2^2) + 1 = 0, \quad \text{and} \quad c_1^2 + c_2^2 - c_1^2 c_2^2 - 1 = 0$$

whose solutions are, up to permutation, $(c_1, c_2) = (1, 2), (1, 0)$. The two solutions are Example 4 on $\mathbb{C}\mathbb{P}^2$ and Example 5 on $\mathbb{S}^2 \times \mathbb{S}^2$.

5.2.2. $a=4$. As in the previous case v_2 and v_3 must be trigonometric functions. From (5.1) it follows that $v_2 + v_3$ is even while $v_2 - v_3$ is odd:

$$v_1 = 4 \sin(t), \quad v_2 = b_1 \cos(ct) + b_2 \sin(ct), \quad v_3 = b_1 \cos(ct) - b_2 \sin(ct)$$

with $b_1, b_2, c > 0$.

If we impose the vanishing of the first few derivatives of the three sectional curvatures at $t = 0$, we obtain the following equations:

$$c^4(3b_1^4 - 2b_1^2 b_2^2 - 9b_2^4) - c^2(b_1^2 b_2^2 + 12b_1^2 - 36b_2^2) + 144 = 0$$

$$c^4(b_1^4 + 2b_1^2 b_2^2 - 3b_2^4) - c^2(b_1^4 + 2b_1^2 b_2^2) + 48 = 0, \quad c^2(b_1^2 - 3b_2^2) - b_1^2 + 12 = 0$$

The last equation implies that there are two possibilities: $b_1^2 = 12$ and $b_2^2 = 4$, or $c^2 = (b_1^2 - 12)/(b_1^2 - 3b_2^2)$. In the former case the first two equations reduce to $4c^2 - 7c^2 + 3 = 0$, $4c^2 - 5c^2 + 1 = 0$ and hence $c = 1$. In the latter case, substituting c^2 into the second equation gives you $3b_1^4(b_2^2 - 4)(b_2^2 + b_1^2 - 16) = 0$. If $b_2^2 = 4$ then the first equation becomes $b_1^4 - 8b_1^2 - 48 = 0$ and hence $b_1^2 = 12$ and thus $c = 1$ again. If $b_2^2 = 16 - b_1^2$, then the first equation gives you $b_1^2 = 8$. Thus $b_2^2 = 8$ and hence $c = 1/2$.

Thus the only solutions are:

$$(b_1, b_2, c) = (2\sqrt{3}, 2, 1) \text{ and } (b_1, b_2, c) = (2\sqrt{2}, 2\sqrt{2}, \frac{1}{2})$$

and we obtain Example 1 on \mathbb{S}^4 , and Example 2 on $\mathbb{C}\mathbb{P}^2$ (up to parametrization).

5.3. $c = 0$. In this case $v_1 = at$ with $a = 2$ or $a = 4$.

5.3.1. $a=2$. As in the previous case, the functions have the form

$$v_1 = 2t, \quad v_2 = b \cosh(c_1 t), \quad v_3 = b \cosh(c_2 t)$$

with $b > 0, c_i \geq 0$ and $c_1 \neq c_2$ which satisfy the equations

$$b^4 c_1^4 - 6b^4 c_1^2 c_2^2 + b^4 c_2^4 - 8b^2 c_1^2 - 8b^2 c_2^2 - 48 = 0$$

$$b^4 c_1^4 - 6b^4 c_1^2 c_2^2 + b^4 c_2^4 - 24b^2 c_1^2 + 24b^2 c_2^2 - 48 = 0$$

$$b^4 c_1^4 - 6b^4 c_1^2 c_2^2 + b^4 c_2^4 + 24b^2 c_1^2 - 24b^2 c_2^2 - 48 = 0$$

This implies $b^2(2c_1^2 - c_2^2) = 0$ and $b^2(c_1^2 - c_2^2) = 0$ and hence there are no solutions.

5.3.2. $a=4$.

$$v_1 = 4t, \quad v_2 = b_1 \cosh(ct) + b_2 \sinh(ct), \quad v_3 = b_1 \cosh(ct) - b_2 \sinh(ct)$$

with $b_1, b_2 > 0, c \geq 0$. Here we get

$$c^4(b_1^4 - 2b_1^2b_2^2 - 3b_2^4) + 48 = 0, \quad c^2(b_1^2 + 3b_2^2) - 12 = 0$$

Solving the second equation for c^4 and substituting into the first implies $b_1 = 0$, contradicting $v_2(0) \neq 0$.

5.4. $c = -1$. In this case $v_1 = a \sinh(t)$ with $a = 2$ or $a = 4$.

5.4.1. $a=2$. The functions must have the form

$$v_1 = 2 \sinh(t), \quad v_2 = b \cosh(c_1 t), \quad v_3 = b \cosh(c_2 t)$$

with $b > 0, c_1, c_2 \geq 0$ and $c_1 \neq c_2$. The vanishing of the derivatives of the curvature gives:

$$b^4 c_1^4 - 6b^4 c_1^2 c_2^2 + b^4 c_2^4 - 8b^2 c_1^2 - 8b^2 c_2^2 - 48 = 0$$

$$b^4 c_1^4 - 6b^4 c_1^2 c_2^2 + b^4 c_2^4 - 4b^4 c_1^2 + 8b^4 c_2^2 - 24b^2 c_1^2 + 24b^2 c_2^2 - 48 = 0$$

$$b^4 c_1^4 - 6b^4 c_1^2 c_2^2 + b^4 c_2^4 + 8b^4 c_1^2 - 4b^4 c_2^2 + 24b^2 c_1^2 - 24b^2 c_2^2 - 48 = 0$$

which easily implies that

$$b^2(c_1^2 - c_2^2)(b^2 + 4) = 0, \quad b^2(c_2^2 - 2c_1^2)(b^2 + 4) = 0,$$

which has no non-zero solutions.

5.4.2. $a=4$. Here the functions are

$$v_1 = 4 \sinh(t), \quad v_2 = b_1 \cosh(ct) + b_2 \sinh(ct), \quad v_3 = b_1 \cosh(ct) - b_2 \sinh(ct)$$

with $b_1, b_2, c > 0$. These functions need to satisfy the equations

$$c^4(3b_1^4 + 2b_1^2b_2^2 - 9b_2^4) + c^2(b_1^2b_2^2 + 12b_1^2 - 36b_2^2) + 144 = 0$$

$$c^2(b_1^2 + 3b_2^2) - b_1^2 - 12 = 0, \quad c^4(8b_1^2b_2^2 - 12b_2^4) + c^2(b_1^2b_2^2 - 48b_1^2 + 48b_2^2) + 12b_1^2 = 0$$

Thus $c^2 = (b_1^2 + 12)/(b_1^2 + 3b_2^2)$ and substituting into the third equation gives you

$$\frac{9b_1^4(b_2^2 - 4)(-b_2^2 + b_1^2 + 16)}{(b_1^2 + 3b_2^2)^2} = 0.$$

If $b_2 = 2$, then $c = 1$ and the first equation becomes $b_1^4 + 8b_1^2 - 48$ which implies $b_1^2 = 4$. If $b_1^2 = b_2^2 - 16$, then the first equation gives you $b_2^2 = 8$ contradicting $b_1^2 > 0$.

Thus we have only one solution, $(b_1, b_2, c) = (2, 2, 1)$, and hence the metric is

$$v_1 = 4 \sinh(t) = 2(e^t - e^{-t}), \quad v_2 = 2 \cosh(t) + 2 \sinh(t) = 2e^t, \quad v_3 = 2 \cosh(t) - 2 \sinh(t) = 2e^{-t}$$

which is the Tsukada's Example 3.

5.4.3. *Codimension 4*. We will finally discuss the case of a codimension 4 singular orbit with the metric still being diagonal. Here smoothness implies that $v_i(0) = 0$ and $v'_i(0) = 1$, $i = 1, 2, 3$. Thus each function is one of

$$\frac{1}{b} \sin(bt), \quad \frac{1}{b} \sinh(bt), \quad t.$$

This gives rise to the following cases.

5.4.4. *Case 1.* If the manifold is compact, we have the following possibility:

$$v_i = \frac{1}{b_i} \sin(b_i t), \quad i = 1, 2, 3$$

which gives rise to the equations

$$5(b_1^2 - b_2^2)b_3^2 - b_1^4 + b_2^4 = 0, \quad 5(b_3^2 - b_1^2)b_2^2 + b_1^4 - b_3^4 = 0$$

One easily sees that the only solutions are $(b_1, b_2, b_3) = (b, b, b)$ and $(b_1, b_2, b_3) = (2b, b, b)$ giving rise to Example 7 and 9, up to parametrization.

5.4.5. *Case 2.* In the second case, we have

$$v_i = \frac{1}{\sqrt{b_i}} \sinh(\sqrt{b_i} t), \quad i = 1, 2, 3$$

The equations and solutions are the same as in the previous case, giving rise to Examples 8 and 10, up to parametrization.

5.4.6. *Case 3.* Combinations of hyperbolic and linear functions have no solutions. For example

$$v_1 = v_2 = \frac{1}{\sqrt{b_i}} \sinh(\sqrt{b_i} t), \quad v_3 = t$$

gives rise to the equations

$$b_1^2 - 5b_2^2 = 0, \quad 5b_1^2 - b_2^2 = 0, \quad 3b_2^4 - 3b_1^4 = 0$$

with only 0 solutions.

The last case is $v_i = t$ which is of course flat euclidean space, Example 11.

6. THE GENERAL CASE.

Recall that if the metric is not necessarily diagonal, then we have two cases. Either H acts trivially on $\mathfrak{n} = \mathfrak{h}^\perp$ and $N(H) = G = SU(2)$, or $\mathfrak{n} = \mathbb{R} \oplus \mathbb{C}$ with H acting trivially on \mathbb{R} and as $-\text{Id}$ on \mathbb{C} and hence $N(H)/H = S^1$. In the first case the metric is arbitrary on \mathfrak{n} , and in the second case \mathbb{R} and \mathbb{C} are orthogonal, but the metric on \mathbb{C} is arbitrary. In either case, there exists an interval $[a, b]$ on the regular part of M such that the multiplicities of P_t are constant. According to Proposition 2.4, the metric is thus equivariantly isometric to one where it is diagonal in $I = (a, b)$. We will assume from now on that this is the case. Let $P_t = (v_1, v_2, v_3)$ be this metric with $v_i = |X_i|$. The curvature tensor is again given by the formulas in (5.1). Thus it follows that if the metric is curvature homogeneous in the interval I , then the three functions v_i are of the form

$$v_i = a_i \sin(d_i t) + b_i \cos(d_i t), \quad v_i = a_i e^{d_i t} + b_i e^{-d_i t}, \quad v_i = a_i t + b_i, \quad t \in I$$

These functions are analytic on all of \mathbb{R} and the curvature functions, according to (5.1), are analytic as long as $v_i > 0$.

Let J be a connected interval such that $I \subset J$, and such that none of the functions v_i vanish on J . Then the functions v_i define an invariant analytic metric on a cohomogeneity one manifold $M' = G/H \times J$, where H is the isotropy group at the regular points of $\gamma(t)$. Although this may not be the original metric on $J \setminus I$, it is still curvature homogeneous. Indeed, the components of the curvature are analytic functions, and since they are constant on I , they must be constant on J as well. We choose J to be maximal with this property and we then have two possible cases:

- (a) $J \subsetneq \mathbb{R}$. This implies that at least one of the functions v_i has a zero at a boundary point t_0 of J . We assume, for simplicity, $t_0 = 0$ and $J = (0, a)$. Moreover one of the functions, say v_1 , takes on (up to scaling) one of the following simplified forms:

$$v_1 = a_1 \sin(t), \quad v_1 = a_1 (e^t - e^{-t}), \quad v_1 = a_1 t^2.$$

The component of the curvature tensor on M' must be constant in J . The vanishing of the derivatives of the curvature tensor will enable us to determine the functions v_i . Notice though that we are not allowed to use the smoothness conditions as we did in Section 5.

- (b) $J = \mathbb{R}$. In this case the functions v_i fall into one of the following types

$$v_i = a_i e^{d_i t} + b_i e^{-d_i t}, \quad v_i = a_i e^{2d_i t}, \quad v_i = a_i$$

where a_i and b_i may be assumed to be positive, and $d_i \geq 0$. In this case it will be sufficient for us to use the fact that the sectional curvature $R(E_2, E_3, E_2, E_3)$ is equal to a constant k . Thus the numerator of $R(E_2, E_3, E_2, E_3) - k$ must be 0.

The result will be that in both cases the functions v_i in the interval J and hence in I , agree with one of the known examples in Section 3. The calculations are significantly more involved than in Section 5. We illustrate the process with one example for each case.

For case (a) let

$$f_1(t) = a_1 t, \quad f_2(t) = a_2 \sin(d_2 t) + b_2 \cos(d_2 t) \quad f_3(t) = a_3 e^{d_3 t} + b_3 e^{-d_3 t},$$

with $a_1, d_2, d_3 \neq 0$. The leading term in the power series of $R(E_2, E_3, E_2, E_3)$ is

$$\frac{(a_3 + b_2 + b_3)^2 (a_3 - b_2 + b_3)^2}{((a_3 + b_3)^2 a_1^2 b_2^2)} t^{-2}.$$

We assume first that $b_2 \neq 0$, which implies that $b_2 = \pm(a_3 + b_3)$. Let $b_2 = (a_3 + b_3)$, the other case being similar. Substituting into $R(E_1, E_3, E_1, E_3)$ and $R(E_1, E_2, E_1, E_2)$, their leading term becomes

$$\frac{(a_1^2 - 8)(a_3 - b_3)d_3 + 8a_2 d_2}{a_1^2 (a_3 + b_3)} t^{-1}, \quad \frac{(a_1^2 - 8)a_2 d_2 + 8d_3(a_3 - b_3)}{a_1^2 (a_3 + b_3)} t^{-1}$$

which easily implies that either $a_2 = 0$ and $a_3 = b_3$, or $a_1 = 4$ and $(a_3 - b_3)d_3 + a_2 d_2 = 0$.

In the first case, substituting into $R(E_3, E_1, E_2, E_4)$ and $R(E_1, E_2, E_3, E_4)$, the first non-constant term is:

$$\frac{-4(d_2^2 + d_3^2)a_3^2 + a_1^2(d_2^2 - 3d_3^2)}{8a_3^2 a_1} t^2, \quad \frac{-4(d_2^2 + d_3^2)a_3^2 + a_1^2(3d_2^2 - d_3^2)}{8a_3^2 a_1} t^2$$

which implies that $d_2 = d_3 = 0$, which is not allowed.

In the second case, $a_1 = 4$ and $(a_3 - b_3)d_3 + a_2 d_2 = 0$, we solve for a_2 and substitute into $R(E_2, E_3, E_1, E_4)$, the first non-constant term is:

$$\frac{3(d_2^2 + d_3^2)(a_3 - b_3)d_3}{2a_3 + 2b_3} t$$

which implies $b_3 = a_3$ and we are back in the first case.

If $b_2 = 0$, and hence $a_2 \neq 0$, then $R(E_2, E_3, E_2, E_3)$ is constant only if $b_3 = -a_3$. Substituting into $R(E_2, E_3, E_1, E_4)$ and $R(E_3, E_1, E_2, E_4)$, the first non-constant term is:

$$\frac{-2a_2^2 d_2^2 + 4a_3^2 d_3^2 + 2a_1^2}{d_2^2 a_1^2 a_2^2} t^{-2}, \quad \frac{2a_2^2 d_2^2 + 4a_3^2 d_3^2 - 2a_1^2}{d_2^2 a_1^2 a_2^2} t^{-2}$$

and hence $a_3 d_3 = 0$. Thus there are no solutions in this case as well.

The case where all 3 functions are trigonometric is more complicated and is left to the reader.

For case (b), let

$$v_1 = a_1 e^{d_1 t} + b_1 e^{-d_1 t}, \quad v_2 = a_2 e^{d_2 t}, \quad v_3 = a_3 e^{d_3 t}.$$

with $a_i, b_i, d_i > 0$. Then $R(E_2, E_3, E_1, E_4)$ is equal to:

$$\frac{-2e^{-t(d_1+d_2+d_3)}}{a_2 a_3 (b_1 + a_1 e^{2d_1 t})} \left(a_2^2 (d_2 - d_3) e^{2t(d_1+d_2)} - a_3^2 (d_2 - d_3) e^{2t(d_1+d_3)} + 2a_1 b_1 (d_2 + d_3) e^{2d_1 t} \right. \\ \left. - a_1^2 (2d_1 - d_2 - d_3) e^{4d_1 t} + (2d_1 + d_2 + d_3) b_1^2 \right).$$

This must be equal to a constant k , and hence the numerator of $R(E_2, E_3, E_1, E_4) - k$ must vanish:

$$-a_1^2 (2d_1 - d_2 - d_3) e^{t(3d_1-d_2-d_3)} + 2a_1 b_1 (d_2 + d_3) e^{t(d_1-d_2-d_3)} - a_3^2 (d_2 - d_3) e^{t(d_1-d_2+d_3)} \\ + a_2^2 (d_2 - d_3) e^{t(d_1+d_2-d_3)} + (2d_1 + d_2 + d_3) b_1^2 e^{-t(d_1+d_2+d_3)} - k a_2 a_3 a_1 e^{2d_1 t} - k a_2 a_3 b_1 = 0.$$

The exponential functions must all cancel and we can use the fact that exponentials with different exponents are linearly independent. Thus we need to distinguish the cases where some coefficient of the exponential functions vanish, or some exponents are equal to 0, or two or more of the exponents coincide. One easily checks that the exponent $d_1 - d_2 - d_3$ cannot be equal to any of the others. Thus it must vanish. Substituting $d_1 = d_2 + d_3$ into the equations, we get

$$- (a_1^2 (d_2 + d_3) + k a_1 a_2 a_3) e^{2t(d_2+d_3)} + 3b_1^2 (d_2 + d_3) e^{-2t(d_2+d_3)} \\ + a_2^2 (d_2 - d_3) e^{2d_2 t} - a_3^2 (d_2 - d_3) e^{2d_3 t} + (2d_2 + 2d_3) a_1 b_1 - k a_2 a_3 b_1 = 0.$$

The coefficient of the second term is positive, and its exponent cannot be equal to any of the other exponents. Thus there are no solutions of this type.

The case where two or all three functions v_i are sums of two exponentials is more involved since the equations will contain many exponential functions. This case is left to the reader.

Thus we have shown that there are no curvature homogeneous metrics when the action of G has no singular orbits.

If there are singular orbits,, we concluded that there exists an interval I on which the metric functions v_i agree with one of the examples in Section 3. The following finishes the proof:

LEMMA 6.1. *Let $(M^4, SU(2))$ be a cohomogeneity one manifold which is curvature homogeneous and assume that the metric is diagonal. If the metric is defined on the interval $[0, L]$ with a singular orbit at $t = 0$ then one of the following holds on $(0, L)$:*

- (a) *The metric is diagonal with 3 distinct eigenvalues and is one of Examples 1-5.*
- (b) *The metric is diagonal with 2 distinct eigenvalues and is one of Examples 6-8.*
- (a) *The metric is diagonal with one eigenvalue and is one of Examples 9-11.*

Proof. (a) We first observe that among the examples in Section 3, only Examples 1-5 have 3 distinct eigenvalues on some interval. If the example has a maximal interval of definition $[0, L]$ (possibly $L = \infty$) one observes that in fact for all $0 < t < L$ the metric has distinct eigenvalues.

Let (a, b) be a maximal connected interval where our metric P_t has 3 distinct eigenvalues. At $t = a$ two of the eigenvalues of P_t coincide by assumption, i.e. there exists a pair, say λ_1, λ_2 such that $\lambda_1(a) = \lambda_2(a)$. We want to show that $a = 0$, so assume that $a > 0$.

There exists a constant $D > 0$ such that for the 5 known examples $P_t^o = \text{diag}(v_1^o, v_2^o, v_3^o)$ we have $|v_i^o(t) - v_j^o(t)| > D$ for $i \neq j$ and $t \in [a, b]$. Now choose a small $\epsilon > 0$ such that $|\lambda_1(t) - \lambda_2(t)| < D/2$ for $t \in (a, a + \epsilon)$. By Proposition 2.4, the metric is equivariantly isometric to one where P_t is diagonal in $(a + \epsilon, b - \epsilon)$. Our proof above shows that on that interval P_t agrees with one of the examples in Section 3. Hence $|v_i^o(a + \epsilon) - v_j^o(a + \epsilon)| = |\lambda_1(a + \epsilon) - \lambda_2(a + \epsilon)| < D/2$. This is a contradiction and hence $a = 0$. Similarly, it follows that $b = L$.

If there is no interval with 3 distinct eigenvalues, we repeat the same argument if there exists an interval with two or one distinct eigenvalues. \square

Remark. In all cases, except for the Tsukada example, we can also argue as follows. The remaining known examples, i.e. the metrics on a symmetric space, are Einstein. After we prove that the metric on the interval I must be one of the known examples, it follows that it is Einstein on I . Being curvature homogeneous implies that it is Einstein on M . This is an ODE along the geodesic, and by the uniqueness of solutions of this ODE starting at a point on I , the metric is isometric to a known example. One can also show that the Tsukada example satisfies an ODE, but it turns out that this is more complicated than the above Lemma.

REFERENCES

- [AA] A.V. Alekseevsky and D.V. Alekseevsky, *G- manifolds with one dimensional orbit space*, Ad. in Sov. Math. **8** (1992), 1–31.
- [AB] M. Alexandrino and R. Bettiol, *Lie Groups and Geometric Aspects of Isometric Actions*, Springer 2015.
- [Be] Besse
- [BKV] Boeckx, O. Kowalski, and L. Vanhecke, *Riemannian manifolds of conullity two*, World Scientific Publishing Co., 1996, pp. xviii+300.
- [Br] T. Brooks, *Three dimensional manifolds with constant Ricci eigenvalues $(\lambda, \lambda, 0)$* , in preparation.
- [FKM] D.Ferus, H.Karcher and H.F. Münzner, *Clifford algebras and new isoparametric hypersurfaces*, Math. Z. **177** (1981), 479-502.
- [GVZ] K. Grove, L. Verdiani and W. Ziller, *An exotic T_1S^4 with positive curvature*, Geom. Funct. Anal. **21** (2011), 499-524.
- [L] P. Lax, *Linear Algebra and Its Applications*, 2nd. ed., Wiley-Interscience, (2007).
- [Se] K. Sekigawa, *On the Riemannian manifolds of the form $B_f \times F^n$* , Kodai Math. Sem. Rep. 26 (1974/75), pp. 343–347.
- [Ta] H. Takagi,, *On curvature homogeneity of Riemannian manifolds*, Tohoku Math.J. 26 (1974), 581-585.
- [Si] I. M. Singer, *Infrinitesimally homogeneous spaces*, Comm. Pure Appl. Math. 13 (1960), pp. 685–697.
- [Ts] K. Tsukada, *Curvature homogeneous hypersurfaces immersed in a real space form*, Tohoku Math. J. 40 (1988), 221–244.
- [TV] F. Tricerri, L. Vanhecke, *Curvature homogeneous Riemannian manifolds*, Ann. scient. Ec. Norm. Sup. **22** (1989), 535–554.
- [KTV] O. Kowalski, F. Tricerri, L. Vanhecke, *Curvature homogeneous Riemannian manifolds*, J. Math. Pures Appl. 71 (1992) 471-501.
- [V] L. Verdiani, *Curvature homogeneous metrics of cohomogeneity one*, Riv. Mat. Univ. Parma **6** (1997), 179–200.
- [VZ] L. Verdiani,W. Ziller, *Smoothness conditions in cohomogeneity one manifolds*, Preprint (2019).
- [Zi] W.Ziller, *On the geometry of cohomogeneity one manifolds with positive curvature*, in: Riemannian Topology and Geometric Structures on Manifolds, in honor of Charles P.Boyer’s 65th birthday, Progress in Mathematics 271 (2009), 233–262.

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