ON THE RICCI ITERATION FOR HOMOGENEOUS METRICS ON SPHERES AND PROJECTIVE SPACES

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Abstract. We study the Ricci iteration for homogeneous metrics on spheres and complex projective spaces. Such metrics can be described in terms of modifying the canonical metric on the fibers of a Hopf fibration. When the fibers of the Hopf fibration are circles or spheres of dimension 2 or 7, we observe that the Ricci iteration as well as all ancient Ricci iterations can be completely described using known results. The remaining and most challenging case is when the fibers are spheres of dimension 3. On the 3-sphere itself, using a result of Hamilton on the prescribed Ricci curvature equation, we establish existence and convergence of the Ricci iteration and confirm in this setting a conjecture on the relationship between ancient Ricci iterations and ancient solutions to the Ricci flow. In higher dimensions we obtain sufficient conditions for the solvability of the prescribed Ricci curvature equation as well as partial results on the behavior of the Ricci iteration.

1. Introduction and main results

Let (M, g_1) be a smooth Riemannian manifold. A Ricci iteration is a sequence of metrics g_i on M satisfying

$$\operatorname{Ric} g_{i+1} = g_i, \quad i \in \mathbb{N},$$

where $\operatorname{Ric} g_{i+1}$ denotes the Ricci curvature of g_{i+1}. This concept was introduced by the third-named author [16, 17] as a discretization of the Ricci flow; see the survey [18, §6.5] and references therein. In order to study a Ricci iteration, first one has to understand when the prescribed Ricci curvature equation has a solution. This is an old subject; see, e.g., [1, Chapter 5], [4, 7] and references therein.

If (M, g_1) is Kähler, a Ricci iteration exists if and only if a positive multiple of g_1 represents the first Chern class, and the iteration then converges modulo diffeomorphisms to a Kähler–Einstein metric whenever one exists [6]. In the non-Kähler case the Ricci iteration was studied only recently for a limited class of homogeneous spaces [15]. The purpose of this article is to add to these results in the non-Kähler case by studying the Ricci iteration for homogeneous metrics on Hopf fibrations. Indeed, such spaces have up to four summands in their isotropy representations with possibly equivalent isotropy summands, while most of the analysis of [15] pertains to two inequivalent summands.

When studying the Ricci flow, following Hamilton [9, §19], it is important to understand the behavior of ancient solutions, i.e., solutions that can be extended indefinitely backward in time. These solutions are the prototype for singularity models for the Ricci flow and have been crucial, for example, in Perelman's work [11]. In our recent work, two of us proposed the following discrete analogue of an ancient Ricci flow [15, §1]: an ancient Ricci iteration is a sequence of Riemannian metrics g_i on M such that

$$g_{i-1} = \operatorname{Ric} g_i, \qquad i = 1, 0, -1, -2, \ldots.$$

The idea is that ancient iterations should "detect" ancient flows, i.e., the latter should exist if the former exist [15, Conjecture 2.5]. It is rare that a Ricci iteration exists since the prescribed curvature problem can often be obstructed. It seems even rarer that an ancient Ricci iteration exists since $\operatorname{Ric} \cdots \operatorname{Ric} g_1$ must always be positive definite. If this is the case, we say that g_1 admits an ancient Ricci iteration. A basic question then is to study convergence and the behavior of the limit since by the aforementioned conjecture this should be helpful for studying ancient Ricci flows.

In this article we verify this conjecture for all homogeneous spheres and complex projective spaces. In the case of homogeneous metrics on $\mathbb{S}^3$, the following result completely describes both Ricci iterations and ancient Ricci iterations, and in particular confirms [15, Conjecture 2.5] in this setting as well.

Notice though, that in general, given T, we can only hope to solve $\operatorname{Ric} g = cT$ for some constant c, see e.g. Theorem 4.1 and Theorems 3.1, 3.2. But in the second iteration step we can solve $\operatorname{Ric} g_2 = g_1$ since Ric is scale invariant. Thus, we say that g_i is a Ricci iteration starting from cg_0 if $\operatorname{Ric} g_1 = cg_0$ and $\operatorname{Ric} g_{i+1} = g_i$ for $i \geq 1$.

Theorem A. *Let g_0 be a left-invariant metric on* $\mathrm{SU}(2)$*.*

(a) *There exists a unique Ricci iteration starting from cg_0 for some $c > 0$, and it converges to a round metric.*
(b) *The only left-invariant metrics which admit an ancient Ricci iteration are the Berger metrics g^λ with $\lambda \in (0, 1]$. Unless $\lambda = 1$, the sequence g_i with $g_1 = g^\lambda$ collapses in the Gromov–Hausdorff topology, as $i \to -\infty$, to a round metric on $\mathbb{S}^2$ by shrinking the length of the Hopf fibers.*

Recall that we have the Hopf fibration $\mathbb{S}^1 \hookrightarrow \mathbb{S}^3(1) \to \mathbb{S}^2(1/2)$ and that a Berger metric g^λ is obtained by changing the length of the fibers to be equal to $2\pi\lambda$. During the collapse the length of the fibers goes monotonically to 0, and hence the metric converges (in the Gromov–Hausdorff topology) to a metric on the base. We point out that one has the same behavior for ancient solutions of the Ricci flow of left-invariant metrics on $\mathbb{S}^3$ [3].

Homogeneous metrics on spheres and complex projective spaces were classified in [19]. Apart from the round sphere and the Fubini–Study metric on complex projective spaces in any real/complex dimension, these metrics can be described geometrically in terms of the Hopf fibrations:

$$\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2n+1} \to \mathbb{CP}^n, \qquad \mathbb{S}^3 \hookrightarrow \mathbb{S}^{4n+3} \to \mathbb{HP}^n, \qquad \mathbb{S}^7 \hookrightarrow \mathbb{S}^{15} \to \mathbb{S}^8,$$
$$\mathbb{S}^2 \hookrightarrow \mathbb{CP}^{2n+1} \to \mathbb{HP}^n.$$

By scaling the canonical metric on the total space by a constant t in the direction of the fibers, and a constant s perpendicular to it, we obtain the homogeneous metrics $g_{t,s}$. The only remaining homogeneous metrics are given by the family $g_{(t_1,t_2,t_3,s)}$ on $\mathbb{S}^{4n+3}$ where we modify the round sphere metric with an arbitrary left invariant metric on the 3-sphere fiber. Up to isometry, one can assume the metric is diagonal with respect to a fixed basis and t_i are the lengths squared of the basis vectors. For the metrics $g_{t,s}$ on $\mathbb{S}^{2n+1}$ and $\mathbb{S}^{15}$, as well as for the metrics $g_{(t,t,t,s)}$ on $\mathbb{S}^{4n+3}$, the isotropy representation consists of two inequivalent irreducible summands, a situation that was studied in detail in [15] where Ricci iterations and ancient Ricci iterations were classified. We summarize the application to the metrics $g_{t,s}$ in Section 3.

The situation for the metrics $g_{(t_1,t_2,t_3,s)}$, which can be regarded as a generalization of the metrics in Theorem A, is more complicated. For the prescribed Ricci curvature problem we have the following sufficient condition.

Theorem B. *Assume that the homogeneous metric $T = g_{(a_1,a_2,a_3,b)}$ satisfies*

$$\frac{b}{a_i} < 2n + 4, \quad i = 1, 2, 3.$$

Then there exists a homogeneous metric g such that $\operatorname{Ric} g = cT$ for some $c > 0$.

In the special case where all a_i are the same, this condition is necessary and sufficient; see Remark 3.3. In Theorem 5.4 we present a strengthened version of the conditions in Theorem B, which according to Figure 1 seems to be necessary and sufficient. Among the metrics $g_{(t_1,t_2,t_3,s)}$ there exists a special subclass with $t_2 = t_3$. These metrics are, in addition, invariant under the rotation of the plane corresponding to the t_2, t_3 variables. We study these metrics in Theorem 5.5.

For the question of the existence of Ricci iterations we obtain the following partial result.

Theorem C. *There exists a neighborhood $\mathcal{O}$ of a round metric h_0 in the space of homogeneous metrics on $\mathbb{S}^{4n+3}$ such that for every $g_0 \in \mathcal{O}$ a Ricci iteration starting with a multiple of g_0 exists and converges to a metric of constant curvature.*

The organization is as follows. Section 2 describes the construction of homogeneous metrics on spheres and projective spaces in terms of Hopf fibrations and homogeneous spaces. Section 3 summarizes the behavior of the Ricci iteration and ancient Ricci iteration for the two-parameter families $g_{t,s}$. Section 4 describes both behaviors for the left-invariant metrics on $\mathrm{SU}(2) \simeq \mathbb{S}^3$ and classifies the ancient Ricci iterations. The proofs of Theorems B and C appear in Sections 5–6. Finally, we include in the Appendix a uniqueness result of independent interest concerning the prescribed Ricci curvature problem within the 4-parameter family of metrics on $\mathbb{S}^{4n+3}$.

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2. Homogeneous metrics on spheres and projective spaces

2.1. **Homogeneous metrics on spheres.** Homogeneous metrics on spheres were classified by the fourth-named author [19], and we now review this classification. There are two ways to visualize such metrics. On the one hand, they can be described via Hopf fibrations with fibers $\mathbb{S}^1, \mathbb{S}^3$, or $\mathbb{S}^7$. On the other hand, they can be realized by classical homogeneous space constructions.

Let us first give the former description. In even dimensions, the only homogeneous metrics on the sphere are the round ones. On odd-dimensional spheres, homogeneous metrics can be described geometrically in terms of Hopf fibrations,

$$\text{(i)} \quad \mathbb{S}^1 \hookrightarrow \mathbb{S}^{2n+1} \to \mathbb{CP}^n, \qquad \text{(ii)} \quad \mathbb{S}^7 \hookrightarrow \mathbb{S}^{15} \to \mathbb{S}^8, \qquad \text{(iii)} \quad \mathbb{S}^3 \hookrightarrow \mathbb{S}^{4n+3} \to \mathbb{HP}^n.$$

Scaling the round metric of curvature 1 on the total space by a constant $t > 0$ in the direction of the fibers, and a constant $s > 0$ in the horizontal direction, we obtain the metrics $g_{t,s}$ in each of the cases (i)–(iii). The only remaining homogeneous metrics are given by the family $g_{(t_1,t_2,t_3,s)}$ on $\mathbb{S}^{4n+3}$ for $n > 0$ (see the beginning of Section 5 for the details) and the family of left-invariant metrics on SU(2).

Let us now relate this to the homogeneous space description; see [15, §2] and references therein for further details. A homogeneous metric is a G-invariant Riemannian metric on the homogeneous space

$$M := G/H,$$

where H is a closed subgroup of a Lie group G. We assume that G is compact and that G and H are connected. Let $\mathfrak{g}, \mathfrak{h}$ denote the Lie algebras of G, H, and let Q be an $\mathrm{Ad}_G(G)$-invariant inner product on $\mathfrak{g}$. The Q-orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ is an $\mathrm{Ad}_G(H)$-invariant subspace of $\mathfrak{g}$, denoted by $\mathfrak{m}$. Thus, $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$, and

$$\begin{aligned} \mathcal{M} &:= \{G\text{-invariant Riemannian metrics on } M\} \\ &\cong \{\mathrm{Ad}_G(H)\text{-invariant inner products on } \mathfrak{m}\}. \end{aligned} \tag{2.1}$$

Consider a Q-orthogonal $\mathrm{Ad}_G(H)$-invariant decomposition

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_q \tag{2.2}$$

such that $\mathrm{Ad}_G(H)|_{\mathfrak{m}_i}$ is irreducible for each $i = 1, \dots, q$. Ignoring the case $q = 1$ (where $\dim \mathcal{M} = 1$) the only homogeneous spheres are as follows [19, p. 352]:

$$\begin{aligned}
q = 2: \quad & \mathbb{S}^{2n+1} = \mathrm{SU}(n+1)/\mathrm{SU}(n) = \mathrm{U}(n+1)/\mathrm{U}(n), \\
& \mathbb{S}^{4n+3} = \mathrm{Sp}(n+1)\mathrm{Sp}(1)/\mathrm{Sp}(n)\mathrm{Sp}(1), \\
& \mathbb{S}^{15} = \mathrm{Spin}(9)/\mathrm{Spin}(7), \\
q = 3: \quad & \mathbb{S}^{4n+3} = \mathrm{Sp}(n+1)\mathrm{U}(1)/\mathrm{Sp}(n)\mathrm{U}(1), \\
& \mathbb{S}^{3} = \mathrm{SU}(2), \\
q = 4: \quad & \mathbb{S}^{4n+3} = \mathrm{Sp}(n+1)/\mathrm{Sp}(n).
\end{aligned}$$

On all of these, we assume Q is induced by the round metric of curvature 1.

Let us explain how the two descriptions are tied. Via (2.1), we can identify $\mathcal{M}$ with a subset of $\mathfrak{m}^* \otimes \mathfrak{m}^*$. The Hopf fibrations can be written in the form

$$K/H \to G/H \to G/K,$$

where K is a subgroup of G containing H. The representation $\mathrm{Ad}_G(H)$ on K/H is 1-, 3- or 7-dimensional if $q = 2$, splits into three 1-dimensional subspaces if $G = \mathrm{SU}(2)$ or $q = 4$, and splits into two irreducible subspaces of dimensions 1 and 2 if $G \neq \mathrm{SU}(2)$ and $q = 3$. Thus, say in the case of $q = 2$,

$$g_{t,s} = t\pi_1^* Q + s\pi_2^* Q,$$

where $\pi_i : \mathfrak{m} \to \mathfrak{m}_i$ denote the natural projections induced by (2.2).

Another useful observation is that the $\mathrm{Sp}(n+1)\mathrm{Sp}(1)$- and $\mathrm{Sp}(n+1)\mathrm{U}(1)$-invariant metrics on $\mathbb{S}^{4n+3}$ are actually all contained in the family $\{g_{(t_1,t_2,t_3,s)}\}$ (see Section 5 for the rigorous definition of this family). The $\mathrm{Sp}(n+1)\mathrm{Sp}(1)$-invariant ones are precisely the metrics $\{g_{(t,t,t,s)}\}$, while the $\mathrm{Sp}(n+1)\mathrm{U}(1)$-invariant ones are precisely the metrics

$$\{g_{(t,t,u,s)}\} \cup \{g_{(t,u,t,s)}\} \cup \{g_{(u,t,t,s)}\}.$$

2.2. **Homogeneous metrics on complex projective spaces.** Once again, we ignore the case $q = 1$. One is then left with the odd-dimensional complex projective spaces $\mathbb{CP}^{2n+1}$ that can be described as $\mathrm{Sp}(n+1)/\mathrm{Sp}(n)\mathrm{U}(1)$, and in this case, $q = 2$ [19, p. 356]. For similar reasons to those in the previous subsection, the 2-parameter space of homogeneous metrics $\{t\pi_1^* Q + s\pi_2^* Q \,:\, t, s > 0\}$, with an appropriate choice of Q, coincides with the family $\{g_{t,s}\}$ obtained from the Hopf fibration $\mathbb{S}^2 \hookrightarrow \mathbb{CP}^{2n+1} \to \mathbb{HP}^n$ described in Section 1.

3. Two isotropy components

In this section we briefly recall how some of our previous work handles the case of circle, 2-sphere and 7-sphere fibers, as well as 3-sphere fibers with additional symmetry. Namely, the cases we consider here are:

$$
\begin{aligned}
\mathbb{S}^{2n+1} &= \mathrm{SU}(n+1)/\mathrm{SU}(n) = \mathrm{U}(n+1)/\mathrm{U}(n),\\
\mathbb{S}^{4n+3} &= \mathrm{Sp}(n+1)\mathrm{Sp}(1)/\mathrm{Sp}(n)\mathrm{Sp}(1),\\
\mathbb{S}^{15} &= \mathrm{Spin}(9)/\mathrm{Spin}(7),\\
\mathbb{CP}^{2n+1} &= \mathrm{Sp}(n+1)/\mathrm{Sp}(n)\mathrm{U}(1).
\end{aligned}
\tag{3.1}
$$

Thus, $(\dim \mathfrak{m}_1, \dim \mathfrak{m}_2) \in \{(1,2n),(3,4n),(7,8),(2,4n)\}$. This means $\dim \mathfrak{m}_1 \neq \dim \mathfrak{m}_2$, so $\mathrm{Ad}_G(H)|_{\mathfrak{m}_1}$ is inequivalent to $\mathrm{Ad}_G(H)|_{\mathfrak{m}_2}$, i.e., the main assumption of [15, Theorems 2.1, 2.4] is satisfied. The other assumption in those theorems is that $Q([X,Y],Z) \neq 0$ for some $X \in \mathfrak{m}_1$ and $Y, Z \in \mathfrak{m}_2$. As explained in [15, §2], this holds unless all metrics in $\mathcal{M}$ have the same Ricci curvature, which is not the case for the spaces (3.1) by the curvature formulas in [19]. Finally, to formulate the conclusion of these theorems we need one more piece of information on the spaces (3.1), namely, the classification of Einstein metrics on them. Denote

$$
\begin{aligned}
\mathcal{E} &:= \{\text{Einstein metrics in } \mathcal{M}\},\\
\alpha_- &:= \inf\big\{t/s \,:\, T = t\pi_1^*Q + s\pi_2^*Q \in \mathcal{E}\big\},\\
\alpha_+ &:= \sup\big\{t/s \,:\, T = t\pi_1^*Q + s\pi_2^*Q \in \mathcal{E}\big\}.
\end{aligned}
$$

Thus, $(\alpha_-, \alpha_+) \in \{(1,1),(1/(2n+3),1),(3/11,1),(1/(n+1),1)\}$ [19]. Given all this, we can now completely describe the Ricci iteration and ancient Ricci iterations on (3.1).

We start with the simplest case of $\mathbb{S}^{2n+1}$ where there is a unique Einstein metric. In this case H is not maximal, but a central extension thereof is, with Lie algebra $\mathfrak{h} \oplus \mathfrak{u}(1)$. The following is a consequence of [15, Theorems 2.1 (ii-a), 2.4 (ii-a)].

Theorem 3.1. *Let $g_{t,s}$ be a homogeneous metric on $\mathbb{S}^{2n+1}$. Then:*

(a) *There exists a unique Ricci iteration starting with $cg_{t,s}$ for some $c > 0$, and it smoothly converges to a round metric.*

(b) *There exists an ancient Ricci iteration starting with $g_{t,s}$ if and only if $t \leq s$. If $t < s$, this iteration converges in the Gromov–Hausdorff topology to a multiple of the Fubini–Study metric on $\mathbb{CP}^n$ by shrinking the fibers to 0, i.e., $t \to 0$.*

In the remaining three cases, H is not maximal and nor does $\mathrm{Ad}_G(H)$ act trivially on $\mathfrak{m}_1$, and there are two Einstein metrics. The following is a consequence of [15, Theorems 2.1 (ii-b), 2.4 (ii-b)].

Theorem 3.2. *Assume that $g_{t,s}$ is a homogeneous metric on $\mathbb{S}^{4n+3}$ with fibers of dimension 3 (or $\mathbb{S}^{15}$ with fibers of dimension 7, or $\mathbb{CP}^{2n+1}$). Then:*

(a) *There exists a Ricci iteration starting with $cg_{t,s}$ for some $c > 0$ if and only if $t/s \geq 1/(2n+3)$ (or $t/s \geq 3/11$, or $t/s \geq 1/(n+1)$). Such a Ricci iteration is unique. Unless $t/s = 1/(2n+3)$ (or $t/s = 3/11$, or $t/s = 1/(n+1)$) such an iteration converges towards a round metric (or a multiple of the standard metric in the case of $\mathbb{CP}^{2n+1}$).*

(b) *There exists an ancient Ricci iteration starting with $g_{t,s}$ if and only if $t/s \leq 1$. If $t/s < 1$, this ancient iteration converges to the non-round Einstein metric in $\mathcal{E}$.*

These results show that the behavior of ancient Ricci iterations in the four cases of this section is the same as for ancient solutions to the Ricci flow [3].

Remark 3.2. One can give an alternative description of Theorems 3.1–3.2 that relies on the Hopf fibrations picture from [19] instead of the languange of homogeneous spaces. For the case of spheres in these theorems, if V and H are the vertical and horizontal space of a Hopf fibration of dimensions d_V and d_H, then $g_{t,s} = t\hat{g}_{|V} + s\hat{g}_{|H}$, where $\hat{g}$ is the metric of curvature 1 on $\mathbb{S}^N$, $N = d_V + d_H$, and for $u \in V, x \in H$,

$$\begin{aligned}
&\operatorname{Ric} g_{t,s}(u,u) = (d_V - 1) + d_H t^2/s^2, \\
&\operatorname{Ric} g_{t,s}(x,x) = d_H + 3d_V - 1 - 2d_V t/s, \qquad \operatorname{Ric} g_{t,s}(u,x) = 0.
\end{aligned}$$

One can then obtain Theorems 3.1–3.2 directly from these formulas by monotonicity arguments as in [15, §4.3–4.4]. Similarly, for the homogeneous metrics on $\mathbb{CP}^{2n+1}$ one can use the Hopf fibration $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^{2n+1} \to \mathbb{HP}^n$, where now for $u \in V, x \in H$,

$$\operatorname{Ric} g_{t,s}(u,u) = 4n + 8 - 4t/s, \quad \operatorname{Ric} g_{t,s}(x,x) = 4nt/s - 4s/t, \quad \operatorname{Ric} g_{t,s}(u,x) = 0.$$

Remark 3.3. In the setting of Theorem 3.2, the condition for solving the equation $\operatorname{Ric} h = cg_{t,s}$ is weaker than the condition for the existence of the Ricci iteration. In fact, a solution exists if and only if $s < (2n+4)t$ (or $s < \frac{14}{3}t$, or $s < (n+2)t$), as a computation starting from the formulas in Remark 3.2 shows; cf. [13, Proposition 3.1]. This demonstrates that the behavior of the Ricci iteration can be different from the behavior of the Ricci flow [5] since in some cases even the first iteration is not possible. Notice though that one also has solutions for some negative values of s, i.e., the prescribed Ricci tensor does not have to be positive definite.

4. Metrics on $\mathbb{S}^3$

In this section, we denote by

$$\mathcal{M} = \mathcal{M}(\mathbb{S}^3) = \mathcal{M}(\mathrm{SU}(2))$$

the set of left invariant metrics on $\mathbb{S}^3 = \mathrm{SU}(2)$. Given a metric $g \in \mathcal{M}$, we can diagonalize g with respect to a basis $\{e_1, e_2, e_3\}$ such that

$$[e_i, e_{i+1}] = 2e_{i+2}, \quad i \in \{1,2,3\}, \tag{4.1}$$

with indices mod 3; this follows from the fact that the automorphism group of $SU(2)$ is $SO(3)$, which therefore acts transitively on the (non-empty) set of bases satisfying (4.1). A more general result about diagonalizing metrics on three-dimensional Lie groups is presented in [10, §4]. Thus

$$g(e_i, e_j) = x_i \delta_{ij} \tag{4.2}$$

for some $x_1, x_2, x_3 > 0$. A computation shows that [8, §6]

$$\operatorname{Ric} g\,(e_i, e_j) = \frac{2(x_i^2 - (x_{i+1} - x_{i+2})^2)}{x_{i+1}x_{i+2}} \delta_{ij} =: r_i \delta_{ij}. \tag{4.3}$$

Thus $\operatorname{Ric} g$ is diagonal with respect to the same basis. The following result by Hamilton [8, Theorem 6.1] implies the existence of Ricci iterations on SU(2).

Theorem 4.1 (Hamilton)**.** *For every metric $T \in \mathcal{M}(\mathbb{S}^3)$, there exists a left-invariant Riemannian metric g, unique up to scaling, such that* $\operatorname{Ric} g = cT$ *for some positive constant c.*

Remark 4.4. In addition to Hamilton's original proof, the existence portion of Theorem 4.1 can be proven by variational methods based on [13, Lemma 2.1]. In fact, one can show that the metric g is the global maximum point of the scalar curvature functional on the set $\mathcal{M}_T(\mathrm{SU}(2)) = \{h \in \mathcal{M}(\mathrm{SU}(2)) \,|\, \mathrm{tr}_h T = 1\}$ [14].

4.1. **Convergence of the Ricci iteration.** We now prove Theorem A (a). The proof relies on a simple monotonicity lemma. To state it, suppose g is a left-invariant metric on SU(2) satisfying (4.2). From (4.3) it follows that:

$$\frac{r_i}{r_j} = \frac{x_{\{i,j\}^c} + (x_i - x_j)}{x_{\{i,j\}^c} - (x_i - x_j)} \cdot \frac{x_i}{x_j}, \tag{4.5}$$

where $\{i,j\}^c := \{1,2,3\} \setminus \{i,j\}$, provided $r_j \neq 0$. This shows:

Lemma 4.2. *Assume the Ricci curvature of the metric g is positive-definite. If*

$$x_i/x_j \geq 1 \quad (< 1),$$

then

$$r_i/r_j \geq x_i/x_j \quad (r_i/r_j < x_i/x_j)$$

for all $i, j \in \{1,2,3\}$.

Given a left-invariant metric g_0 on SU(2), Theorem 4.1 implies that there exists a left-invariant metric g_1, unique up to scaling, such that $\operatorname{Ric} g_1 = cg_0$ for some $c > 0$. Applying Theorem 4.1 again, we find that we can scale g_1 uniquely to find g_2, unique up to scaling, so that $Ric(g_2) = g_1$. Repeated application of this argument implies the existence and the uniqueness of the sequence $\{g_i\}_{i\in\mathbb{N}}$. In fact, g_i are all diagonal with respect to $\{e_1, e_2, e_3\}$. Thus:

$$g_i(e_k, e_l) = x_k^{(i)} \delta_{kl}, \quad k \in \{1,2,3\},\ i \in \mathbb{N},$$

with $x_k^{(i)} > 0$. Set

$$\alpha_{kl}^{(i)} := x_k^{(i)}/x_l^{(i)}.$$

Lemma 4.2 implies that if $\alpha_{kl}^{(1)} \geq 1$ (< 1) for some k, l, then $\alpha_{kl}^{(i)} \geq 1$ (< 1) for all $i \in \mathbb{N}$, and the sequence $\{\alpha_{kl}^{(i)}\}_{i\in\mathbb{N}}$ is monotone decreasing (increasing). Thus, for fixed k, l, the sequence $\{\alpha_{kl}^{(i)}\}_{i\in\mathbb{N}}$ converges to some $\alpha_{kl} > 0$. By (4.3),

(4.6)

$$\alpha_{kl}^{(i)} = \frac{x_k^{(i)}}{x_l^{(i)}} = \frac{x_k^{(i+1)}}{x_l^{(i+1)}} \cdot \frac{x_{\{k,l\}^c}^{(i+1)} + x_k^{(i+1)} - x_l^{(i+1)}}{x_{\{k,l\}^c}^{(i+1)} - x_k^{(i+1)} + x_l^{(i+1)}} = \alpha_{kl}^{(i+1)} \frac{\alpha_{\{k,l\}^c k}^{(i+1)} + 1 - \alpha_{lk}^{(i+1)}}{\alpha_{\{k,l\}^c k}^{(i+1)} - 1 + \alpha_{lk}^{(i+1)}}, \qquad i \in \mathbb{N}.$$

Passing to the limit,

$$\alpha_{kl}\big(\alpha_{\{k,l\}^c k} - 1 + \alpha_{lk}\big) = \alpha_{kl}\big(\alpha_{\{k,l\}^c k} + 1 - \alpha_{lk}\big),$$

whence $\alpha_{lk} = 1$. By (4.3),

$$\begin{aligned} x_k^{(i)} &= 2\frac{\big(x_k^{(i+1)} + x_{k+2}^{(i+1)} - x_{k+1}^{(i+1)}\big)\big(x_k^{(i+1)} + x_{k+1}^{(i+1)} - x_{k+2}^{(i+1)}\big)}{x_{k+1}^{(i+1)} x_{k+2}^{(i+1)}} \\ &= 2\big(\alpha_{kk+1}^{(i+1)} + \alpha_{k+2k+1}^{(i+1)} - 1\big)\big(\alpha_{kk+2}^{(i+1)} + \alpha_{k+1k+2}^{(i+1)} - 1\big). \end{aligned} \tag{4.7}$$

Passing to the limit again,

$$\lim_{i\to\infty} x_k^{(i)} = 2,$$

so $\{g_i\}_{i\in\mathbb{N}}$ converges to a round metric on $\mathbb{S}^3$.

4.2. **Classification of ancient Ricci iterations.** Recall that a Berger metric is a metric as in (4.2), with respect to some basis satisfying (4.1), and with $(x_1, x_2, x_3) = (\nu, 2, 2)$. If we let $\mathcal{B}$ be the set of all such bases, then we denote by g_D^ν the Berger metric with respect to the basis $D \in \mathcal{B}$. Notice that one obtains a round metric for $\nu = 2$.

In order to prove Theorem A (b), we can assume that the ancient Ricci iteration $g_{i-1} = \operatorname{Ric} g_i$ for $g_i \in \mathcal{M}(\mathrm{SU}(2))$ has the property that all g_i are diagonal with respect to a fixed basis $\{e_1, e_2, e_3\}$.

Lemma 4.3. *The Berger metric g_D^ν admits an ancient Ricci iteration if and only if $\nu \in (0, 2]$.*

Proof. Let $g_1 = g_D^\nu$. First, suppose $\nu \in (0, 2)$. It suffices to show that

$$x_k^{(i)} > 0 \quad \text{for all } k \in \{1,2,3\} \text{ and } i \le 1. \tag{4.8}$$

Assume by induction that $0 < x_1^{(j)} < 2 \le x_2^{(j)} = x_3^{(j)}$ for all $j \in \{1, 0, \ldots, i+1\}$. We claim this holds also for $j = i$, which then, of course, implies (4.8). Indeed, by (4.3),

$$x_1^{(i)} = 2\big(x_1^{(i+1)}/x_2^{(i+1)}\big)^2, \quad x_2^{(i)} = x_3^{(i)} = 2\big(2x_2^{(i+1)} - x_1^{(i+1)}\big)/x_2^{(i+1)}.$$

Evidently then $x_1^{(i)} > 0$ but also by induction we see that $x_1^{(i)} < 2 < x_2^{(i)} = x_3^{(i)}$.

Next, we assume that $\nu > 2$ and that g_D^ν admits an ancient Ricci iteration. Then the argument of the previous paragraph shows that $0 < x_2^{(i)} = x_3^{(i)} \le 2 < x_1^{(i)}$ for all $i \le 1$. So, $\alpha_{23}^{(i)} = \alpha_{23} = 1$. Also, $\alpha_{21}^{(i)} = \alpha_{31}^{(i)} < 1$, and by Lemma 4.2 the sequences $\{\alpha_{21}^{(i)}\}_{i\in\mathbb{N}}$ and $\{\alpha_{31}^{(i)}\}_{i\in\mathbb{N}}$ are monotone decreasing. The limits of these sequences are in $[0, 1)$. Suppose $\alpha_{21} > 0$. Passing to the limit in (4.6), we obtain $\alpha_{21} = \alpha_{21}\frac{\alpha_{32}+1-\alpha_{12}}{\alpha_{32}-1+\alpha_{12}}$, which implies that $\alpha_{12} = 1$. Hence $\alpha_{21} = 1$, a contradiction. We conclude that $\alpha_{21} = 0$ and $\lim_{i\to-\infty} \alpha_{12}^{(i)} = \infty$. However, going back to (4.2), we get

$$x_2^{(i)} = x_3^{(i)} = 2(2x_2^{(i+1)} - x_1^{(i+1)})/x_2^{(i+1)} = 4 - 2\alpha_{12}^{(i+1)}. \tag{4.9}$$

This quantity is negative for i sufficiently close to $-\infty$. Since $x_2^{(i)}$ is the component of a Riemannian metric, we obtain a contradiction. $\square$

If $g_1 = g_D^\nu$ with $\nu \in (0,2)$, the arguments from the proof of Lemma 4.3 show that $\alpha_{12} = 0$. Similarly, $\alpha_{13} = 0$. Going back to (4.7),

$$x_1^{(i)} = 2\big(\alpha_{12}^{(i+1)} + \alpha_{32}^{(i+1)} - 1\big)\big(\alpha_{13}^{(i+1)} + \alpha_{23}^{(i+1)} - 1\big) = 2\alpha_{12}^{(i+1)}\alpha_{13}^{(i+1)}$$

yields $\lim_{i\to-\infty} x_1^{(i)} = 0$. Going back to (4.9), we get

$$\lim_{i\to-\infty} x_2^{(i)} = \lim_{i\to-\infty} x_3^{(i)} = 4.$$

Thus, the $\mathbb{S}^1$ fibers of the Hopf fibration collapse, and g_i converge in the Gromov–Hausdorff topology to a round metric on $\mathbb{S}^2$.

The next lemma completes the classification of homogeneous ancient Ricci iterations.

Lemma 4.4. *The metric $g \in \mathcal{M}$ admits an ancient Ricci iteration if and only if $g = cg_D^\nu$ for some $c > 0$, $\nu \in (0,2]$ and $D \in \mathcal{B}$.*

Proof. Let g be such that $g \neq cg_D^\nu$ for any $c, \nu > 0$ and $D \in \mathcal{B}$. Assume that there exists an ancient Ricci iteration starting with $g = g_1$. Since $g \neq cg_D^\nu$, we can assume that

$$x_1^{(1)} < x_2^{(1)} < x_3^{(1)}. \tag{4.10}$$

Lemma 4.2 and (4.10) imply that $\alpha_{21}^{(i)}, \alpha_{31}^{(i)}$, and $\alpha_{32}^{(i)}$ are all monotonically increasing. We claim that they all converge. Indeed, by assumption, $x_k^{(i)} > 0$ and $Ric(g_i)$ is positive-definite for all $i \leq 1$, so by (4.5), $\frac{x_{\{k,l\}^c}^{(i)} + x_k^{(i)} - x_l^{(i)}}{x_{\{k,l\}^c}^{(i)} - x_k^{(i)} + x_l^{(i)}} > 0$ for all k, l. Therefore, $x_{\{k,l\}^c}^{(i)} + x_k^{(i)} - x_l^{(i)} > 0$ for all k, l, so

$$\alpha_{32}^{(i)} < \alpha_{12}^{(i)} + 1.$$

The right-hand side is monotonically decreasing, while the left-hand side is monotonically increasing. Thus, both sides converge, and since $\alpha_{32} > \alpha_{32}^{(1)} > 1$, we get $\alpha_{12} > 0$. Also, $\alpha_{31} = \alpha_{32}/\alpha_{21} > 0$. However, passing to the limit in (4.6), we obtain $\alpha_{21} = \alpha_{21}\frac{\alpha_{32}+1-\alpha_{12}}{\alpha_{32}-1+\alpha_{12}}$, which gives $\alpha_{12} = 1$. This is a contradiction, as $\alpha_{12} < \alpha_{12}^{(1)} < 1$. □

5. Four-parameter family of metrics

We now discuss the homogeneous metrics on $\mathbb{S}^{4n+3} = \mathrm{Sp}(n+1)/\mathrm{Sp}(n)$. If H is the horizontal space, then the gauge group $\mathrm{Sp}(1) = N(H)/H$ acts on the vertical space V as $\mathrm{SO}(3)$ via the twofold cover $\mathrm{Sp}(1) \to \mathrm{SO}(3)$ [19, p. 353]. It thus acts transitively on the set of (oriented) bases orthonormal with respect to the metric on V induced by the round metric $\hat{g}$ of curvature 1. Hence, given any $\mathrm{Sp}(n+1)$-invariant metric g, we can assume that there exists a basis $\{e_1, e_2, e_3\}$ of V satisfying (4.1) in which g is diagonal, i.e., there are some positive constants x_1, x_2, x_3, s so that

$$g(e_i, e_j) = x_i\delta_{ij}, \quad g_{|H} = s\,\hat{g} \quad \text{and} \quad g(e_i, u) = 0$$

for all horizontal vector fields u. We denote this metric by $g = g_{(x_1,x_2,x_3,s)}$.

The Ricci curvature of g satisfies, for all horizontal vector fields u,

$$\begin{aligned} \operatorname{Ric} g(e_i, e_j) &= \left(4n\frac{x_i^2}{s^2} + 2\frac{x_i^2 - (x_{i+1} - x_{i+2})^2}{x_{i+1}x_{i+2}}\right)\delta_{ij}, \\ \operatorname{Ric} g(u, u) &= \left(4n + 8 - 2\frac{x_1 + x_2 + x_3}{s}\right)g(u, u), \\ \operatorname{Ric} g(e_i, u) &= 0, \end{aligned}$$

and is thus again diagonal with respect to the same basis.

We now study the question of prescribing the Ricci curvature. Let T be a metric invariant under $\mathrm{Sp}(n+1)$. We want to solve $\operatorname{Ric} g = \kappa T$, $\kappa > 0$, for a homogeneous metric g. Assuming T is diagonal in some basis $\{e_1, e_2, e_3\}$ of V, we set

$$T(e_i, e_j) = T_i\delta_{ij}, \quad T_{|H} = b\,\hat{g}. \tag{5.1}$$

We find the following sufficient condition:

Theorem 5.1. *Assume the* $\mathrm{Sp}(n+1)$*-invariant metric* T *on* $\mathbb{S}^{4n+3}$ *satisfies*

$$\frac{b}{T_i} < 2n + 4, \quad i = 1, 2, 3. \tag{5.2}$$

Then there exists an $\mathrm{Sp}(n+1)$*-invariant metric* g *such that* $\operatorname{Ric} g = \kappa T$ *for some* $\kappa > 0$*.*

Remark 5.3. According to Remark 3.3, the equation $\operatorname{Ric} g = \kappa g_{a,b}$ has a solution if and only if $b/a < 2n + 4$. The above theorem generalises the "if" part of this statement.

Proof. It is sufficient to prove the claim if $b = 1$. We will prove the existence of a metric $g = g_{(x_1,x_2,x_3,s)}$, diagonal in the basis $\{e_1, e_2, e_3\}$, with Ricci curvature κT. Since Ric is scale-invariant, we can assume that $s = 1$. Consider the following system of equations depending on a parameter λ:

$$\begin{aligned} c &= (4n + 8) - 2(x_1 + x_2 + x_3), \\ x_2x_3cT_1 &= \lambda x_1^2x_2x_3 + 2(x_1^2 - (x_2 - x_3)^2), \\ x_1x_3cT_2 &= \lambda x_2^2x_1x_3 + 2(x_2^2 - (x_1 - x_3)^2), \\ x_1x_2cT_3 &= \lambda x_3^2x_2x_1 + 2(x_3^2 - (x_1 - x_2)^2). \end{aligned} \tag{5.4}$$

If $\lambda = 4n$, a solution to the system gives us the desired solution to $\operatorname{Ric} g = \kappa T$ with $\kappa = c$. By varying λ from 0 to $4n$, we will show that (5.4) has a solution for all $\lambda \in [0, 4n]$ using degree theory. See pages 185–225 in [2] for an introduction to the subject of degree theory.

Notice that if $\lambda = 0$, the last three equations

$$\begin{aligned} x_2x_3cT_1 &= 2(x_1^2 - (x_2 - x_3)^2), \\ x_1x_3cT_2 &= 2(x_2^2 - (x_1 - x_3)^2), \\ x_1x_2cT_3 &= 2(x_3^2 - (x_1 - x_2)^2) \end{aligned} \tag{5.5}$$

are the prescribed Ricci curvature equations for a left-invariant metric on $\mathrm{SU}(2)$. Recall that Hamilton's Theorem 4.1 says that we can solve equations (5.5), and that the solution

(x_1, x_2, x_3, c) is unique up to scaling of (x_1, x_2, x_3). By examining his proof one easily sees that this solution depends differentiably on T_i.

In order to solve the system with $\lambda \neq 0$ we first obtain the following bound:

Lemma 5.2. *If (x_1, x_2, x_3, c) solve the system (5.5) and $1/T_i < 2n+4$ for all i, then $c = c(T_1, T_2, T_3) < 4n+8$.*

Proof. By dividing each of the equations in (5.5) by $x_1x_2x_3$ and adding, we obtain:

$$\begin{aligned} c\sum_{i=1}^{3}\frac{T_i}{x_i} &= 4\sum_{i=1}^{3}\frac{1}{x_i} - 2\frac{x_1}{x_2x_3} - 2\frac{x_2}{x_1x_3} - 2\frac{x_3}{x_1x_2} \\ &= 4\sum_{i=1}^{3}\frac{1}{x_i} - \frac{1}{x_1}\Big(\frac{x_2}{x_3}+\frac{x_3}{x_2}\Big) - \frac{1}{x_2}\Big(\frac{x_1}{x_3}+\frac{x_3}{x_1}\Big) - \frac{1}{x_3}\Big(\frac{x_1}{x_2}+\frac{x_2}{x_1}\Big) \\ &\leq 4\sum_{i=1}^{3}\frac{1}{x_i} - 2\sum_{i=1}^{3}\frac{1}{x_i} = 2\sum_{i=1}^{3}\frac{1}{x_i} \end{aligned}$$

since $a + 1/a \geq 2$ for $a > 0$. This implies the claim. $\square$

In order to apply degree theory, we now show that the set of solutions of (5.4) with $(x_1, x_2, x_3, c) \in (0,\infty)^4$ and $\lambda \in [0, 4n]$ is compact.

Lemma 5.3. *There exists an open pre-compact convex subset Ω of $(0,\infty)^4$ such that for $\lambda \in [0, 4n]$, any solution (x_1, x_2, x_3, c) of (5.4) lies in Ω.*

Proof. Assume to the contrary that no such set exists. Then there is a sequence of $\lambda^{(i)} \in [0, 4n]$ with a corresponding sequence $\big(x_1^{(i)}, x_2^{(i)}, x_3^{(i)}, c^{(i)}\big) \in (0,\infty)^4$ of solutions to (5.4) such that one of the variables goes to 0 or ∞. For the remainder of the proof, we surpress reference to i to simplify notation. The first equation in (5.4) shows that no variable can go to ∞. We will consider two cases, first that $c \to 0$, and second, that at least one of x_1, x_2 or x_3 goes to 0 and c is does not converge to 0. We show that we get a contradiction in both cases.

First Case. If $c \to 0$, then passing to the limits of (5.4), we find that $x_1 \to y_1$, $x_2 \to y_2$, $x_3 \to y_3$ and $\lambda \to \mu$, where y_i and μ are non-negative numbers solving

$$\begin{aligned} 0 &= (4n+8) - 2(y_1+y_2+y_3), \\ 0 &= \mu y_1^2 y_2 y_3 + 2(y_1^2 - (y_2-y_3)^2), \\ 0 &= \mu y_2^2 y_1 y_3 + 2(y_2^2 - (y_1-y_3)^2), \\ 0 &= \mu y_3^2 y_1 y_2 + 2(y_3^2 - (y_1-y_2)^2). \end{aligned} \tag{5.6}$$

First, we claim that at least two of y_1, y_2, y_3 are identical. To see this, note that by taking differences of the last three equations of (5.6), we find

$$\begin{aligned} 0 &= (y_1-y_2)\left(\mu y_1y_2y_3 + 4(y_1+y_2-y_3)\right), \\ 0 &= (y_2-y_3)\left(\mu y_1y_2y_3 + 4(y_2+y_3-y_1)\right), \\ 0 &= (y_3-y_1)\left(\mu y_1y_2y_3 + 4(y_1+y_3-y_2)\right). \end{aligned} \tag{5.7}$$

If all of y_1, y_2, y_3 are distinct, then at least one of y_1, y_2, y_3 is positive, and (5.7) implies that

$$\begin{aligned} 0 &= (\mu y_1 y_2 y_3 + 4(y_1 + y_2 - y_3))\,, \\ 0 &= (\mu y_1 y_2 y_3 + 4(y_2 + y_3 - y_1))\,, \\ 0 &= (\mu y_1 y_2 y_3 + 4(y_1 + y_3 - y_2))\,. \end{aligned} \tag{5.8}$$

By adding these three equations up, we find that the non-negative numbers y_1, y_2, y_3 and μ satisfy $3\mu y_1 y_2 y_3 + 4(y_1 + y_2 + y_3) = 0$, which is a contradiction since one of y_1, y_2, y_3 is positive.

We now know that at least two of y_1, y_2, y_3 are identical, so we assume without loss of generality that $y_2 = y_3$. Then since μ, y_1, y_2 and y_3 are all non-negative, the second equation of (5.6) implies that $y_1 = 0$, and the first equation implies that $y_2 = y_3 > 0$. Now by dividing the third and fourth equations of (5.4) by $x_1 x_3$ and $x_1 x_2$ respectively, we see that

$$\begin{aligned} cT_2 &= \lambda x_2^2 + 2\Big(\frac{x_2^2}{x_1 x_3} - \frac{x_1}{x_3} - \frac{x_3}{x_1} + 2\Big), \\ cT_3 &= \lambda x_3^2 + 2\Big(\frac{x_3^2}{x_1 x_2} - \frac{x_1}{x_2} - \frac{x_2}{x_1} + 2\Big). \end{aligned}$$

Since $x_1 \to y_1 = 0$, $c \to 0$, $\lambda \geq 0$ and $x_2, x_3 \to y_2 = y_3 > 0$, we deduce that eventually, both $\frac{x_2^2}{x_1 x_3} - \frac{x_3}{x_1}$ and $\frac{x_3^2}{x_1 x_2} - \frac{x_2}{x_1}$ must be negative. Thus $x_2^2 - x_3^2 < 0$ and $x_3^2 - x_2^2 < 0$, which is a contradiction.

Second Case. Now c converges to some positive number, but at least one of x_1, x_2, x_3 is converging to 0. To start, assume that $x_1 \to 0$. The third equation of (5.4) implies that $x_2 - x_3 \to 0$. The second equation of (5.4) then implies that $x_2 x_3 c \to 0$. Since c does not converge to 0, we must have both x_2 and x_3 converging to 0 as well as x_1. If instead of assuming $x_1 \to 0$ we assume that $x_2 \to 0$ or $x_3 \to 0$, we would again conclude that all three of x_1, x_2, x_3 are converging to 0.

Since $x_1, x_2, x_3 \to 0$, the first equation of (5.4) implies that $c \to 4n + 8$. Rewrite the second, third and fourth equations as

$$cT_1 = \lambda x_1^2 + 2z_2 z_3, \qquad cT_2 = \lambda x_2^2 + 2z_1 z_3, \qquad cT_3 = \lambda x_3^2 + 2z_1 z_2, \tag{5.9}$$

where $z_1 = \frac{x_2 + x_3 - x_1}{x_1}$, $z_2 = \frac{x_1 + x_3 - x_2}{x_2}$, $z_3 = \frac{x_1 + x_2 - x_3}{x_3}$, and we can assume that these numbers change monotonically as well. For each $i = 1, 2, 3$, $\lambda x_i^2 \to 0$ and $cT_i \to (4n + 8)T_i > 0$, so (5.9) implies that all three of z_1, z_2, z_3 are bounded, from which we deduce that $\frac{x_i}{x_j}$ is bounded for each i and j. Now write (5.9) as

$$
\begin{aligned}
cT_1 &= \lambda x_1^2 + \frac{2((dx_1)^2 - (dx_2 - dx_3)^2)}{dx_2 dx_3}, \\
cT_2 &= \lambda x_2^2 + \frac{2((dx_2)^2 - (dx_1 - dx_3)^2)}{dx_1 dx_3}, \\
cT_3 &= \lambda x_3^2 + \frac{2((dx_3)^2 - (dx_1 - dx_2)^2)}{dx_2 dx_1},
\end{aligned}
\tag{5.10}
$$

where $d = 1/x_1$ (again dropping reference to the superscript). By taking a subsequence, we can assume that dx_2 and dx_3 are monotone. Since $\frac{x_i}{x_j}$ is a bounded sequence for each i and j, we know that dx_2 and dx_3 converge to some positive numbers. Taking limits and using the fact that $c \to 4n+8$, we see that the numbers dx_i converge to a solution of (5.5) with $c = 4n+8$. However, this contradicts Lemma 5.2. $\square$

Define a smooth function $f_\lambda \colon (0,\infty)^4 \to \mathbb{R}^4$, $\lambda \in [0, 4n]$ with

$$f_\lambda(x_1, x_2, x_3, c) = (a_0, a_1, a_2, a_3),$$

where

$$
\begin{aligned}
a_0 &= c + 2(x_1 + x_2 + x_3), \\
a_1 &= \lambda \frac{x_1^2}{c} + \frac{2(x_1^2 - (x_2 - x_3)^2)}{c x_2 x_3}, \\
a_2 &= \lambda \frac{x_2^2}{c} + \frac{2(x_2^2 - (x_1 - x_3)^2)}{c x_1 x_3}, \\
a_3 &= \lambda \frac{x_3^2}{c} + \frac{2(x_3^2 - (x_2 - x_2)^2)}{c x_1 x_2}.
\end{aligned}
$$

Then

$$f_\lambda(x_1, x_2, x_3, c) = (4n+8, T_1, T_2, T_3) := y$$

is equivalent to (5.4). We want to show that $f_\lambda^{-1}(y)$ is nonempty for all $\lambda \in [0, 4n]$, which implies our theorem when $\lambda = 4n$. We first show that this holds when $\lambda = 0$. Recall that by using Theorem 4.1, we can solve the last 3 equations in (5.4), which coincide with equations (5.5) when $\lambda = 0$, and that the solution is unique up to scaling. Lemma 5.2 implies that under the assumption $1/T_i < 2n+4$, $i = 1, 2, 3$, we can choose this scaling so that the first equation in (5.4) is satisfied as well. Thus $f_0^{-1}(y)$ consists of a single point $p \in \Omega$, where Ω is some open, pre-compact convex set satisfying the conclusion of Lemma 5.3 (Lemma 5.3 implies that such sets exist). Since the solution depends differentiably on T_i, it follows that f_0^{-1} is differentiable near y and hence $\det(Df_0)_p \neq 0$, which implies that $deg(f_0|_\Omega, y) = \pm 1$, by Definition 2 on page 186 of [2]. Notice that since Ω satisfies the conclusion of Lemma 5.3, all the solutions of (5.4) are in the interior of Ω, so $y \notin f_\lambda(\partial\Omega)$. Thus by the homotopy invariance of the Brouwer degree (see Theorem 3 (iv) on page 190 of [2]), it follows that $deg(f_\lambda|_\Omega, y) \neq 0$ for all λ. Therefore, by Theorem 3 (i) on page 190 of [2], $f_\lambda^{-1}(y) \cap \Omega$ is non-empty. This finishes the proof. $\square$

The condition in Theorem 5.1 is not necessary. In fact, its proof shows that one has the following stronger statement:

Theorem 5.4. *There exists an* $\mathrm{Sp}(n+1)$*-invariant metric* g *such that* $\operatorname{Ric} g = \kappa T$ *for some* $\kappa > 0$ *if one has*

$$c = c(T_1/b, T_2/b, T_3/b) < 4n + 8, \tag{5.11}$$

where $c(x_1, x_2, x_3)$ *is defined in terms of the solution to the system* (5.5).

It is easy to find an explicit expression for the function c. In fact,

$$c(x_1, x_2, x_3) = 8\frac{x_1 Z^2 - x_3}{x_1^2 Z^2 - x_3^2},$$

where Z is a solution of the cubic equation

$$x_1^2(x_2 - x_3)Z^3 + x_1x_3(2x_1 - x_2 - x_3)Z^2 + x_1x_3(2x_3 - x_1 - x_2)Z + x_3^2(x_2 - x_1) = 0.$$

As was pointed out to us by Renato Bettiol, one can implement the so called cylindrical algebraic decomposition algorithm, not to solve the equations $\operatorname{Ric} g = \kappa T$ explicitly, but to decide whether a solution exists for a given T. In Figure 1 the red box indicates the set of T such that the existence of a solution is guaranteed by Theorem B. Here we assume that $n = 1$, and the tensor T satisfying (5.1) with $b = 1$ corresponds to the point $(x, y, z) = (\frac{1}{T_1}, \frac{1}{T_2}, \frac{1}{T_3})$. The cylindrical algebraic decomposition algorithm was implemented for every point on a lattice, and the blue dots mark those T for which it predicted the existence of a solution. The cloud of blue dots agrees with the set obtained from Theorem 5.4, suggesting that (5.11) is the necessary and sufficient condition for solvability.

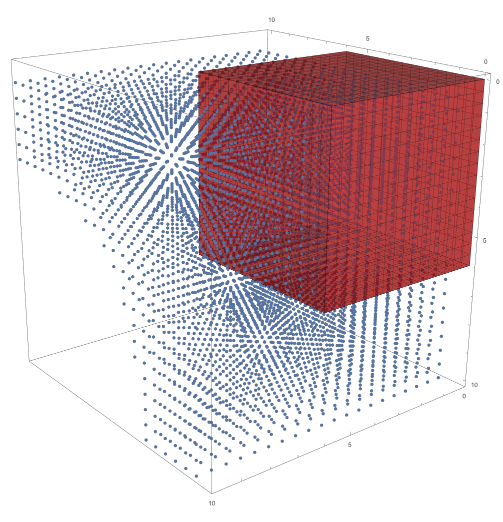

Figure 1. The set of solutions.

Among the four-parameter family of metrics $g_{(x_1,x_2,x_3,s)}$ there exists a special subclass with $x_2 = x_3 = x$. These metrics are invariant under the larger isometry group $\mathrm{Sp}(n+1)\mathrm{U}(1)$, where $\mathrm{U}(1)$ acts by rotation in the plane corresponding to the x_2, x_3 variables. If we restrict our attention to the $\mathrm{Sp}(n+1)\mathrm{U}(1)$-invariant case, we can obtain a simple formula for the function c appearing in (5.4).

Theorem 5.5. *Assume that the metric T on $\mathbb{S}^{4n+3}$ given by (5.1) is $\mathrm{Sp}(n+1)\mathrm{U}(1)$-invariant, i.e., $T_2 = T_3$. If*

$$bT_1 + 4bT_2 - b\sqrt{T_1^2 + 8T_1T_2} < (4n+8)T_2^2,$$

then there exists an $\mathrm{Sp}(n+1)\mathrm{U}(1)$-invariant metric g such that $\mathrm{Ric}\, g = \kappa T$ for some $\kappa > 0$.

Proof. Straightforward verification shows that

$$(x_1, x_2, x_3, c) = \Big(\frac{1}{2T_2}\Big(-T_1 + \sqrt{T_1^2 + 8T_1T_2}\Big)x, x, x, \frac{b}{T_2^2}\Big(T_1 + 4T_2 - \sqrt{T_1^2 + 8T_1T_2}\Big)\Big)$$

is a solution to (5.5) with T_i replaced by T_i/b for every $x > 0$. The claim follows from this observation and Theorem 5.4. $\square$

Remark 5.12. This shows that the condition (5.11) is substantially weaker than (5.2). For example, suppose that $b = 1$ and $T_2 = T_3 > \frac{1}{n+2}$. In this case,

$$c(T_1/b, T_2/b, T_3/b) = \frac{b}{T_2^2}\Big(T_1 + 4T_2 - \sqrt{T_1^2 + 8T_1T_2}\Big) \leq \frac{4}{T_2} < 4n + 8,$$

which means (5.4) holds for all $T_1 > 0$. However, (5.2) only holds for $T_1 > \frac{1}{2n+4}$.

Remark 5.13. If an $\mathrm{Sp}(n+1)\mathrm{U}(1)$-invariant metric $g_{(x_1,x,x,s)}$ admits an ancient iteration, then the following necessary condition holds: $x_1 \leq x \leq s$. The proof of this fact requires careful analysis of monotone quantities associated with the iteration.

Remark 5.14. For homogeneous metrics the Ricci flow can be interpreted as the gradient flow of the negative scalar curvature restricted to the space of metrics of volume 1. The round metric is a local maximum of this functional, and the non-round Einstein metric a saddle point. It has two negative eigenvalues and one positive eigenvalue whose eigenvector is $x_1 = x_2 = x_3$. Thus, at least near the non-round Einstein metric, there exist precisely two ancient solutions, already contained in the family $g_{t,s}$; see [5].

6. Ricci iteration near the round metric

The general behavior of Ricci iterations among the family $g_{(x_1,x_2,x_3,s)}$ seems to be difficult to understand. But one can descibe their behavior near a round metric.

The set of homogeneous metrics on $\mathbb{S}^{4n+3}$ has a natural topology since it is determined by the inner products on the tangent space at one point, which in turn form an open cone in the vector space of symmetric bilinear forms on the tangent space. We let h_0 be the round metric $g_{(1,1,1,1)}$.

Theorem 6.1. *There exists a neighborhood $\mathcal{O}$ of the round metric h_0 in the space of homogeneous metrics on $\mathbb{S}^{4n+3}$ such that the following holds: if $g_0 \in \mathcal{O}$, then there exists a Ricci iteration that starts with cg_0 for some $c > 0$ and converges to a metric of constant curvature.*

Proof. Choose Ω to be some open subset of

$$(0,\infty)^3 \setminus \{(x_1,x_2,x_3) \in (0,\infty)^3 : x_1+x_2+x_3 \neq 2n+4\},$$

and let $f = (f_1,f_2,f_3) : \Omega \to \mathbb{R}^3$ be given by

$$f_i(x_1,x_2,x_3) = \frac{4nx_i^2 + 2\frac{(x_i+x_{i+1}-x_{i+2})(x_i+x_{i+2}-x_{i+1})}{x_{i+1}x_{i+2}}}{(4n+8)-2(x_1+x_2+x_3)}.$$

Equations (5.4) imply that if $y_i = f_i(x_1,x_2,x_3)$, then $\operatorname{Ric} g_{(x_1,x_2,x_3,1)} = cT_{(y_1,y_2,y_3,1)}$, where we have $c = (4n+8)-2(x_1+x_2+x_3)$. Therefore, if we have a sequence

$$\left\{x^{(k)} = \left(x_1^{(k)},x_2^{(k)},x_3^{(k)}\right)\right\}_{k=1}^{\infty} \in (0,\infty)^3$$

such that $x^{(k)} = f(x^{(k+1)})$ for $k \geq 1$, then there exist some $c^{(k)} > 0$ such that the metrics

$$g_k = g_{(c^{(k)}x_1^{(k)},c^{(k)}x_2^{(k)},c^{(k)}x_3^{(k)},c^{(k)})} \tag{6.1}$$

form a Ricci iteration.

Next, observe that the derivative of f at $(1,1,1)$ is $2+\frac{1}{2n+1}$ times the identity. Therefore, there exist neighborhoods Ω,Ω' of $(1,1,1)$ such that $f : \Omega \to \Omega'$ is a diffeomorphism. Since $|Df^{-1}| < 1$ at $(1,1,1)$, we can assume, by making Ω' smaller, that $f^{-1}(\Omega') \subset \Omega'$ and $|Df^{-1}| \leq a < 1$ on all of Ω'. Thus f^{-1} is a contraction and hence $x^{(k+1)} = f^{-1}(x^{(k)})$ is a sequence that converges to $(1,1,1)$.

Let $\mathcal{O}$ be any neighborhood of h_0 in the space of homogeneous metrics such that every metric in $\mathcal{O}$ can be transformed by a gauge transformation into a metric of the form $g_{(x_1,x_2,x_3,s)}$ with $(x_1/s,x_2/s,x_3/s) \in \Omega'$. Fix a metric $g_0 \in \mathcal{O}$. Without loss of generality, assume g_0 is diagonal. Then (6.1) defines a sequence of metrics with $\operatorname{Ric} g_{k+1} = g_k$ for $k \geq 1$ and $\operatorname{Ric} g_1 = cg_0$ for some $c > 0$. Furthermore, $\left(x_1^{(k)},x_2^{(k)},x_3^{(k)}\right)$ converges to $(1,1,1)$ and since $c^{(k)} = 4n+8-2\left(x_1^{(k+1)}+x_2^{(k+1)}+x_3^{(k+1)}\right)$, it follows that g_k converges to a metric of constant curvature. $\square$

Remark 6.2. The theorem shows that the round metric is stable for the Ricci iteration. In fact, it is also stable for the Ricci flow since it is a local maximum of the scalar curvature functional on the set of metrics of volume 1.

Appendix A

If a tensor T on $\mathbb{S}^{4n+3}$ is invariant under $\mathrm{Sp}(n+1)\mathrm{Sp}(1)$, it is natural to ask whether every $\mathrm{Sp}(n+1)$-invariant solution to $\operatorname{Ric} g = cT$ must also be $\mathrm{Sp}(n+1)\mathrm{Sp}(1)$-invariant. The following theorem shows that this is indeed the case. As we explained in Section 5, every $\mathrm{Sp}(n+1)$-invariant metric on $\mathbb{S}^{4n+3}$ can be written as $g_{(x_1,x_2,x_3,s)}$ for some $x_1,x_2,x_3,s > 0$. By definition, $g_{(x_1,x_2,x_3,s)}$ is obtained by modifying a round metric $\hat{g}$ with a left-invariant metric on the 3-dimensional fibers of the Hopf fibration and scaling in the horizontal direction. If $x_1 = x_2 = x_3 = t$, then $g_{(x_1,x_2,x_3,s)}$ coincides with the $\mathrm{Sp}(n+1)\mathrm{Sp}(1)$-invariant metric $g_{t,s}$ obtained by scaling $\hat{g}$ by t in the direction of the fibers and by s in the perpendicular direction.

Theorem A.1. *Let g be an $\mathrm{Sp}(n+1)$-invariant metric on $\mathbb{S}^{4n+3}$ with $\mathrm{Sp}(n+1)\mathrm{Sp}(1)$-invariant Ricci curvature. Then, up to isometry, $g = g_{t,s}$ for some t, s.*

Proof. Assume $g = g_{(x_1,x_2,x_3,s)}$. By the scale-invariance of the Ricci curvature, it suffices to consider the case where $s = 1$. Our goal is to show that $x_1 = x_2 = x_3$.

Since $\mathrm{Ric}\, g$ is $\mathrm{Sp}(n+1)\mathrm{Sp}(1)$-invariant, it can be obtained by multiplying $\hat{g}$ by some $a \in \mathbb{R}$ in the direction of the fibers and by some $b \in \mathbb{R}$ in the perpendicular direction. Computing $\mathrm{Ric}\, g$ as in Section 5, we find

$$\begin{aligned} b &= 4n + 8 - 2(x_1 + x_2 + x_3), \\ a &= 4nx_i^2 + 2\frac{x_i^2 - (x_{i+1} - x_{i+2})^2}{x_{i+1}x_{i+2}}, \qquad i = 1, 2, 3. \end{aligned}$$

Let us multiply the second line by $x_{i+1}x_{i+2}$ and take differences of the equations for different i. We see that

$$\begin{aligned} 0 &= (x_1 - x_2)(4nx_1x_2x_3 + 4(x_1 + x_2 - x_3) + ax_3), \\ 0 &= (x_2 - x_3)(4nx_1x_2x_3 + 4(x_2 + x_3 - x_1) + ax_1). \end{aligned} \tag{A.1}$$

First assume that x_1, x_2, x_3 are all distinct. Then

$$\begin{aligned} 0 &= 4nx_1x_2x_3 + 4(x_1 + x_2 - x_3) + ax_3, \\ 0 &= 4nx_1x_2x_3 + 4(x_2 + x_3 - x_1) + ax_1. \end{aligned}$$

Taking the difference of these equalities, we see that $8(x_1 - x_3) + a(x_3 - x_1) = 0$ and $a = 8$, so

$$4nx_1x_2x_3 + 4(x_1 + x_2 - x_3) + ax_3 = 4nx_1x_2x_3 + 4(x_1 + x_2 + x_3) > 0,$$

a contradiction. Thus at least two of x_1, x_2, x_3 are identical, and in this case it is a simple matter to conclude from (A.1) that all three must in fact be identical. □

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