

Cartier divisors

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1 Examples

Example 1.1. Let X be the affine quadric cone $X = \text{Spec } k[X, Y, Z]/(XY - Z^2)$. We will show $\text{CaCl} = 0$ and $\text{Cl} = \mathbb{Z}/2$.

Example 1.2. Let X be a double cover of \mathbb{P}^3 ramified in the Barth sextic surface:

$$4(\tau^2 x^2 - y^2)(\tau^2 y^2 - z^2)(\tau^2 z^2 - x^2) - \omega^2(1 + 2\tau)(x^2 + y^2 + z^2 - w^2)^2 = 0$$

in \mathbb{P}^4 . Here $\tau = \frac{1+\sqrt{5}}{2}$. This is a rational threefold. $\text{CaCl}(X) = \mathbb{Z}$, $\text{Cl}(X) = \mathbb{Z}^{14}$. See example 1.4 in [CJ].

Remark 1.3. For a normal variety, if an integer multiple of a Weil divisor is a Cartier divisor, we call it a \mathbb{Q} -Cartier divisor. We can define operations on Cartier divisors and extend by linearity to \mathbb{Q} -Cartier divisors. (up to numerical equivalence?). If on a normal scheme every Weil divisor is \mathbb{Q} -Cartier, we call it \mathbb{Q} -factorial. It is easy to see Example 1.2 is not \mathbb{Q} -factorial. See Definition 2.12 [McK]. \diamond

Remark 1.4. It is often easy to do computations in toric varieties. See Section 3.3 in [F]. (elaborate?) \diamond

2 Effective divisors

Definition 2.1. Effective divisors.

For a Weil divisor $D = \sum_i a_i D_i$ where D_i are prime Weil divisors, if all the nonzero $a_i > 0$, we call it an effective Weil divisor.

For a Cartier divisor on $X = \cup_i U_i$, if $f_i \in \mathcal{O}(U_i) \cap \mathcal{K}^*(U_i)$, then we say it is an effective divisor.

Check these two definitions are compatible when a Weil divisor is also a Cartier divisor.

Next in the relative situation $X \in \mathfrak{Sch}_S$ we will define relative effective Cartier divisors. Recall the definition of flat morphisms.

Definition 2.2. A morphism $f : X \rightarrow S$ with $f(x) = s$ is said to be flat at x if $f_s^\# : \mathcal{O}_{S,s} \rightarrow \mathcal{O}_{X,x}$ is a flat ring homomorphism. f is said to be a flat morphism if it is flat at every point $x \in X$.

Now suppose $f : X \rightarrow S$ is a flat morphism, $D \subset X$ is an effective Cartier divisor, we say it is also a relative divisor if the following two equivalent conditions are satisfied:

- $f|_D : D \rightarrow S$ is flat.
- $\forall x \in D$, f_x is not a zero divisor in the fibre $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} k_s$.

See Definition 1.11 from [JK]. Instead of proving the above two conditions are equivalent, we will simply state a stronger result (Lemma 9.4.3 from [FAG].)

Now D is still an effective Cartier divisor on X , if we don't require X is flat over S a priori, we still have:

Lemma 2.3. *TFAE:*

- D is flat at x .
- X and D are flat at x , and the fiber D_s is an effective divisor in $X_s := X \otimes_{\mathcal{O}_{S,s}} k_s$.
- X is flat at x , and f_x is not a zero divisor in the fibre $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} k_s$.

Example 2.4. $S = \mathbb{Z}$, $X = \mathbb{Z}[Y]$ (in particular is a UFD.) The divisor given by the height 1 prime ideal (p) , where p is a prime number in \mathbb{Z} , is not relative effective. (Why?) The height 1 prime ideal $(f(Y))$ where $f(Y)$ is a monic irreducible polynomial gives a relative effective divisor. (Why?) Note however although we start from a Cartier divisor which is also a prime Weil divisor, the fibre $f(Y) \bmod p$ doesn't have to be irreducible.

Now let $X \in \mathfrak{Sch}_S$ be flat over the base S , we define a functor $CDiv_{X/S}$ from \mathfrak{Sch}_S to the category of *Set*:

Definition 2.5. $\forall T \in \mathfrak{Sch}_S$,
 $CDiv_{X/S}(T) = \{\text{relative effective divisors on } X_T/T\}$.

Remark 2.6. That this is indeed a functor is essentially due to the fact that tensor product is right exact and keeps flatness. (Why?) \diamond

Recall the definition of representable functors. A contravariant functor \mathcal{F} from $\text{Cat } \mathcal{A}$ to *Set* is representable if there is a natural isomorphism from \mathcal{F} to $h_B := \text{Hom}_{\mathcal{A}}(-, B)$. B is an object in \mathcal{A} that represents the functor \mathcal{F} . This means there is an universal object $U \in \mathcal{F}(B)$, such that $\forall Z \in \text{Cat } \mathcal{A}$, $\text{Hom}_{\mathcal{A}}(Z, B) \rightarrow \mathcal{F}(Z)$ given by $g \rightarrow g^*U$ is an isomorphism.

Theorem 2.7. *Let S be locally noetherian, $X \in \mathfrak{Sch}_S$ be flat and projective. Then*

- $CDiv_{X/S}$ is representable by an open subscheme of $\text{Hilb}_{X/S}$. (we will abuse notation and denote the scheme represents it also $CDiv_{X/S}$.)
- If X/S is smooth, then $CDiv_{X/S} \hookrightarrow \text{Hilb}_{X/S}$ is universally closed.

See Theorem 1.13 in [JK].

(Theorem 1.4 in [JK]. Let X/S be a projective scheme, $\mathcal{O}(1)$ a relatively ample line bundle, P a polynomial. The functor $\text{Hilb}_P(X/S)$ is representable by $U \in X \times_S \text{Hilb}_p(X/S)$. $\text{Hilb}_P(X/S)$ is projective over S .)

3 Sheaf associated to a Cartier divisor

First recall several definition:

Sheaf of \mathcal{O}_X module. $\mathcal{F}(U)$ is an \mathcal{O}_X module for any open U , and for $V \subset U$, $(af)|_V = a|_V f|_V$.

Let \mathcal{F} be an \mathcal{O}_X module, \mathcal{F} is quasi coherent if $\forall x \in X$, there is an open neighborhood U s.t. there is an exact sequence of \mathcal{O}_X modules

$$\mathcal{O}_X^J|_U \rightarrow \mathcal{O}_X^I|_U \rightarrow \mathcal{F}|_U \rightarrow 0$$

Characterization: Let X be a scheme, and \mathcal{F} an \mathcal{O}_X module. Then \mathcal{F} is quasi coherent iff \forall affine open U , $\mathcal{F}(U)^\sim = \mathcal{F}|_U$. (Liu Theorem 5.1.7.)

For an arbitrary coherent sheaf on a Noetherian integral scheme, we can define its rank to be the rank of K vector space at the generic point.

Quasi coherent sheaves form a monoid under tensor product. The identity element being \mathcal{O}_X . We call the invertible elements invertible sheaves. These are the locally free sheaves of rank 1. (Exercise). For a locally free sheaf of rank 1 \mathcal{G} , its inverse is $\mathcal{G}^\vee := \text{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{O}_X)$.

For a Cartier divisor D , represented by (U_i, f_i) as above. We can define the subsheaf of \mathcal{K} : for each U_i , form the sub $\mathcal{O}_X|_{U_i}$ module of \mathcal{K} as $(f_i^{-1}\mathcal{O}_X(U_i))^\sim$. (check well defined.) This is locally free of rank 1 (and it is easy to see it is invertible.) Denoted by $\mathcal{L}(D)$.

$\mathcal{L}(D)$ can also be characterized by $\mathcal{L}(D)(U) = \{f \in \mathcal{K}^*, \text{div}(f) + D|_U > 0\} \cup 0$. (Why?) (This definition easily generalize to the sheaf associated with a Weil divisor.)

Description: D is an effective Cartier/Weil divisor iff $\mathcal{O}_X \subset \mathcal{L}(D)$ as subsheaves of \mathcal{K} .

Theorem 3.1. a) $D \rightarrow \mathcal{L}(D)$ gives a one-one correspondence between Cartier divisors on X and invertible subsheaves of \mathcal{K} .

b)

$$\mathcal{L}(D_1 - D_2) \cong \mathcal{L}(D_1) \otimes \mathcal{L}(D_2)^{-1}$$

c) $D_1 \sim D_2$ iff $\mathcal{L}(D_1) \cong \mathcal{L}(D_2)$ as abstract sheaves, regardless of embedding in \mathcal{K} .

Proof: a) One direction is obvious. For the other direction, D can be recovered from $\mathcal{L}(D)$ by taking f_i to be the inverse of a local generator of $\mathcal{L}(D)|_{U_i}$.

b)

$$\mathcal{L}(D_1 - D_2) = \mathcal{L}(D_1) \cdot \mathcal{L}(D_2) \cong \mathcal{L}(D_1) \otimes \mathcal{L}(D_2)$$

c) By b), it will suffice if $\mathcal{L}(D) \cong \mathcal{O}_X$ iff D is principal. If D is principal, sending $1 \rightarrow f^{-1}$ gives $\mathcal{O}_X \cong \mathcal{L}(D)$. Conversely also. Q.E.D.

Proposition 3.2. When X is integral, every torsion free sheaf of rank 1 is isomorphic to a subsheaf of \mathcal{K} .

In the following we assume X is Noetherian integral.

Definition 3.3. A coherent sheaf \mathcal{F} is called reflexive if the natural map $\mathcal{F}^{\vee\vee}$ is an isomorphism.

Proposition 3.4. 1. $\mathcal{F} \hookrightarrow \mathcal{F}^{\vee\vee}$ is an embedding if \mathcal{F} is torsion free.
 2. For an arbitrary coherent \mathcal{F} , \mathcal{F}^{\vee} is always reflexive.

Theorem 3.5. Let \mathcal{L} be a reflexive subsheaf of \mathcal{K} on a normal variety X , TFAE:
 a) \mathcal{L} is rank 1.
 b) $\mathcal{L} = \mathcal{L}(D)$ for some Weil divisor D .

The definitions and proof here from Chapter 8 in [CoX] and also [KS].

The next few propositions will help us characterize the relation between Weil divisors and Cartier divisors on a normal variety.

Proposition 3.6. Let X be normal Noetherian, (not necessarily quasi projective), \mathcal{F} coherent, then \mathcal{F} is S_2 iff reflexive.

Proposition 3.7. Let X be Noetherian, \mathcal{F} is S_2 coherent on X . $Y \subset X$ be a closed subset of $\text{codim}(Y/X) \geq 2$, $U := X \setminus Y$, then $\mathcal{F}(X) \rightarrow \mathcal{F}(U)$ is an isomorphism.

Corollary 3.8. X normal Noetherian, \mathcal{F} is S_2 , $\mathcal{F} \cong i_*\mathcal{F}|_U$.

On the level of divisors, we have a Weil divisor D on $X \Rightarrow D|_U$ is a Cartier divisor on U . For a Cartier divisor E on U , its Zariski closure \bar{D} is a Weil divisor on X . We already see what happens on the level of their associated sheaves above. This is a one-one correspondence.

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