SOME GEOMETRIC CONSTRUCTION TECHNIQUES IN POLYTOPES

ZHANG YING

0357060

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Professor Dan Zaffran

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1. Introduction

A convex polytope is the convex hull of a finite set of points in the Euclidean space. When its affine hull is of dimension d, we call it a d-polytope. In a d-polytope P we denote the number of j-dimension face as $f_j(P)$, and call $f(P) = (f_0(P), f_1(P), ..., f_{d-1}(P))$ the f-vector of P. By convention we may add $f_{-1}(P) = 1$.

The f-vector is an integer vector. For many years it is not known whether we can find a characterization of the set of f-vectors of all polytopes. And if yes, what is the meaning of it. This is an open problem in the general sense. In 1971 McMullen proposed a characterization of the f-vectors of a special type of polytopes, called simplicial polytopes. In 1979 Billera and Lee proved the sufficiency of his conjecture using an elegant geometric construction. Their techniques are further developed by Kalai to obtain a sharp lower bound for the number of triangulation of spheres (with labelled vertices). In 1980 Stanley proved the necessity of McMullen's conjecture. In this expository paper we will first explain the ideas in Billera and Lee's work. Then we will move on to Kalai's lower bound for the number of triangulated spheres, and look at some properties of his construction.

2. Preliminaries

2.1. **Simplicial polytopes.**

In a d-polytope, we call a $(d-1)$ dimensional face a *facet*, and a 0-dimensional face (one point) a *vertex*. If a polytope is spanned by $d+1$ affinely independent vertices, we call it a *d-simplex*. Any face of a simplex is itself a simplex. Fixing the number of vertices, a simplex is the polytope with the 'most' number of faces, as any $j+1$ ($j < d$) vertices will generate a j-face. So, any two facets intersect with each other, and the intersection will always be a $(d-2)$-face.

A simplicial polytope is less restrictive than a simplex. We call a polytope *simplicial* when all its facets are simplices.

2.2. **Simplicial complex.**

A *complex* is a finite collection of polytopes that contains all the faces of its polytopes and that the intersection of two of its polytopes is a face in each of them. We define the f-vector of a complex in the obvious way, and call its inclusion-maximal polytopes facets. The dimension of the complex is the largest dimension of its facets. A Complex is *pure* if all the facets are of the same dimension.

We call a complex *simplicial* if every facet is a simplex. For example, the boundary complex of a simplicial polytope is naturally a simplicial complex. Sometimes a simplicial complex is abstractly represented as a collection of point sets. We denote a facet by the set of its vertices, and every subset of this set is also in the collection. Conversely, it can be shown that every abstract simplicial complex has a geometric realization. In the following we identify the abstract representation of a simplicial complex with its geometric version.

2.3. ***h*-vectors.** For a $(d-1)$-dimensional simplicial complex Δ, we define its f-polynomial and h-polynomial in the following way: Let

$$f(\Delta, t) = \sum_{j=-1}^{d-1} f_j t^{t+1}, \tag{2.1}$$

$$h(\Delta, t) = \sum_{i=0}^{d} h_i t^i = (1-t)^d f(\frac{t}{1-t}) \tag{2.2}$$

the series of coefficient in the h-polynomial is called the h-vector $(h_0, h_1, ..., h_d)$, or explicitly expressed as

$$h_i = \sum_{j=0}^{i} (-1)^{i-j} \left(\begin{array}{c} d-j \\ d-i \end{array} \right) f_{j-1}, \qquad 0 \le i \le d, \tag{2.3}$$

$$f_j = \sum_{i=0}^{j+1} \left(\begin{array}{c} d-i \\ d-j-1 \end{array} \right) h_i, \qquad -1 \le j \le d-1 \tag{2.4}$$

Particularly, $f_0 = h_1 + d, h_0 = 1, f_{d-1} = \sum_{i=0}^{d} h_i$.

When we mention the h-vector of a simplicial polytope we actually mean its boundary complex. If h is the h-vector of some simplicial polytope, we write $h \in h(\mathscr{P}_s^d)$, where $\mathscr{P}_s^d$ is the collection of all simplicial polytopes. In [6] there is a simple proof that h_i satisfy the Dehn-Sommerville equation:

$$h_i = h_{d-i}, \qquad 0 \le i \le n, n = \lceil d/2 \rceil. \tag{2.5}$$

This equation can be generalized for h-vectors of all simplicial spheres, though the proof method is different from the case of polytopes.

We define the g-vector of a simplicial complex as $g_0 = 1$, $g_i = h_i - h_{i-1}$, $1 \le i \le n$. When the Dehn-Sommerville equations hold there is a bijection between the f-vector and the g- vector: $f = g \centerdot M$, some properties of the matrix M is analyzed in [4].

2.4. **Pseudopower relations.** For any given positive integers h and i, h can be written uniquely in the form

$$h = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \ldots + \binom{n_j}{j}, \tag{2.6}$$

(2.7)

where $n_i > n_{i-1} > \ldots > n_j \geq j \geq 1$. Then define

$$h^{<i>} = \binom{n_i + 1}{i+1} + \binom{n_{i-1}+1}{i} + \ldots + \binom{n_j + 1}{j+1},$$

called the ith pseudopower of h.

2.5. **Shelling.** For a polytope F we write $\overline{F}$ as the set of all its subsets, which is then a complex. We say a pure d-simplicial complex Δ is shellable if its facets can be ordered $F_1, F_2..., F_\omega$ so that for $2 \leq k \leq \omega$, if we put $\Delta_k := \cup_{i=1}^{k} \overline{F_i}$, then $\overline{F_k} \cap \Delta_k = \cup_{j=1}^{s_k} \overline{G_j^k}$, where G_j^k are distinct $(d-1)$ sets of Δ and $s_k \geq 1$. By convention we put $s_1 = 0$.

An equivalent statement for shelling of simplicial complex is: for every $j < k$, $F_j \cap F_k \subseteq F_i \cap F_k$, $i < k$ and $F_i \cap F_k$ is a facet of F_k.

2.6. **Order.** Let $\Phi^{(n)}$ denote the set of all monomials in the variables $\{Y_1, Y_2, \ldots, Y_u\}$ with degree no more than n. Put $Y_0 = 1$. For an monomial $m \in \Phi^{(n)}$, m can be extended uniquely in the form

$$Y_{e_1} Y_{e_2} \ldots Y_{e_n}, \quad 0 \leq e_1 \leq e_2 \leq \ldots \leq e_n \leq u.$$

If we fix an order on the indeterminates Y_{e_i} as indicated by their index, we can then define a reverse lexicographic order on the monomials: $m = Y_{e_1} Y_{e_2} \ldots Y_{e_n} < m' = Y_{e'_1} Y_{e'_2} \ldots Y_{e'_n}$ iff for some k, $Y_{e_k} < Y_{e'_k}$ and for all j, $k < j \leq n$, $Y_{e_k} = Y_{e'_k}$. It is a total order on $\Phi^{(n)}$, and can be defined on other collection of n-tuples (with elements listed in non-decreasing order) as well.

For two monomials m and m', define $m <^* m'$ if $m < m'$ and deg $m \leq$ deg m'. Given a set of monomials M, we call it an order ideal of monomials (or OIM for short) iff it contains every devisor of its members.

2.7. **Cyclic polytope.** The cyclic polytope is an important type of polytopes with many applications. Let $c(t) = (t, t^2, t^3 ... t^{d+1})$ be the moment curve in $\mathbb{R}^{d+1}$, select $\nu > d+1$ distinct points $v_i = c(t_i), t_1 < t_2 < \ldots < t_\nu$. Denote $V = \{v_1, v_2, ..., v_\nu\}$. Then $C(\nu, d+1) = conv\ V$ is a simplicial polytope with the vertices exactly V. Moreover, it can be shown that the face lattice of $C(\nu, d+1)$ is independent of the choices of t_i. (see e.g. [10][18])

Let W be a proper subset of V. For $W' \subseteq W$, $W' = \{v_i, v_i+1..., v_j\}$, $i < j$ we follow the notations in [3] and say W' is a contiguous subset of W if

$$i > 1; \quad j < \nu; \quad v_{i-1},\ v_{j+1} \notin W.$$

When $i = 1$ and $v_{j+1} \notin W$, W' is called the left end-set. When $j = \nu$, $v_{j+1} \notin W$, W' is the right end-set. In the case W' is a contiguous set or right end-set, we call v_{i-1} the antecedent of W'. Of course W can be written uniquely as $W_1 \cup W_2 \cup \ldots \cup W_q$, where W_1, W_q are end-sets (possibly empty), the rest are contiguous sets.

With these notations we can have a characterization of all facets in $C(\nu, d+1)$ (Gale's evenness criterion): W is a facet iff $|W| = d+1$ and all contiguous subsets in W (not including end-sets) are even.

3. Sufficiency of McMullen's Conditions

3.1. **The g-conjecture.** This was the once conjectured characterization of h-vectors of a simplicial polytope, now known as the g-theorem:

Theorem 3.1. *$h \in h(\mathscr{P}_s^d)$ iff the following three conditions hold:*

$$h_i = h_{d-i}, \qquad 0 \le i \le n, \tag{3.1}$$

$$h_{i+1} \ge h_i, \qquad 0 \le i \le n-1, \tag{3.2}$$

$$h_0 = 1, \quad h_{i+1} - h_i \le (h_i - h_{i-1})^{<i>}, \quad 1 \le i \le n-1. \tag{3.3}$$

When the h-vector satisfies McMullen's condition we call the corresponding g-vector an M-sequence.

Next we will proceed an outline of the proof of sufficiency of the theorem. First we will establish a Lemma casting some light on the geometric meaning of h-vectors. The h-vector is connected with the shelling of the complex, yet the relationship is invariant with specific shelling order:

Lemma 3.2. *Let Δ be a shellable $(d-1)$-simplicial complex, $h(\Delta_1, t) = 1, h(\Delta_k, t) = h(\Delta_{k-1}, t) + t^{s_k}$ for $2 \le k \le \omega$, where Δ_k are defined as in subsection 2.5. Hence $h_i(\Delta) = |\{k : s_k = i\}|$, for $0 \le i \le d$.*

Proof. We use induction on dimension $d \ge 1$. For $d = 1$, the facets are distinct points. From the second points on, the intersection with the union of previous points will always be the empty set, which by definition is a (trivial) face in Δ. So for $k \ge 2, s_k = 1$, which suggests $h(\Delta, t) = 1 + (f_0 - 1)t$. The assertion is true.

Now, for $d > 1$, the f-vectors of Δ_k and Δ_{k-1} have the following relationship:

$$f(\Delta_k, t) = f(\Delta_{k-1}, t) + f(\overline{F_k}, t) - f(\cup_{j=1}^{s_k} \overline{G_j^k}, t) \tag{3.4}$$

By equation 2.2, we deduce

$$h(\Delta_k, t) = h(\Delta_{k-1}, t) + h(\overline{F_k}, t) - (1-t)h(\cup_{j=1}^{s_k} \overline{G_j^k}, t) \tag{3.5}$$

Now, as F_k is a simplex, $\partial \overline{F_k}$ is a $(d-2)$-complex with facets shellable in any order. (Next we write Δ_1 for $\overline{F_k}$, as they are combinatorially the same.) Fixing a shelling order in Δ_1, for the kth facet F'_k, we have $s'_k = k-1, 1 \leq k \leq d$. By the induction on dimension,

$$h(\cup_{j=1}^{k} \overline{F'_j}, t) = \sum_{i=0}^{k-1} t^i, \qquad 1 \leq k \leq d. \tag{3.6}$$

We count the faces in Δ_1, (this time including F_1.) Obviously

$$f_k = \begin{pmatrix} d \\ k+1 \end{pmatrix}.$$

So

$$f(\Delta_1, t) = (1+t)^d \Rightarrow h(\Delta_1, t) = 1. \tag{3.7}$$

In equation 3.5, we get

$$h(\Delta_k, t) = h(\Delta_{k-1}, t) + 1 - (1-t)\sum_{i=0}^{s_k - 1} t^i = h(\Delta_{k-1}, t) + t^{s_k}$$

$\square$

Remark. There is a geometric interpretation of the h-vectors in [18]. Note that the h-vectors are defined in such a way that we can also link it with the *Betti* number of some manifold (induced by the simplicial polytope) and the *Hilbert function* of some graded algebra. See [6][13][16].

Now with Lemma 3.2, we can construct a simplicial complex whose h-vector is a given M-sequence (and with redundant zeroes).

First, consider the case $h_1 = 1$. By condition 3.2, 3.3 in Theorem 3.1, $h_2 = h_3 = \ldots = h_d = 1$. Observe directly this is the h-vector of (the boundary of) a d-simplex. Next we will assume $h_1 > 1$.

Let $g = (g_0, g_1, g_2, \ldots, g_n)$ be an M-sequence. Define $\nu' = g_1 + 2n$, $U = \{u_1, u_2, \ldots, u_{\nu'}\}$. When d is odd, define $V' = \{v_1, v_2\}$, and when d is even $V' = \{v_1\}$. Let $V = V' \cup U$ be $\nu = d + g_1 + 1$ distinct points with increasing time t on the $(d+1)$ dimensional moment curve. $C(\nu, d+1) = conv\ V$. F' be a $2n$-subset of U in the form:

$$F' = \{u_{i_1}, u_{i_1+1}\} \cup \{u_{i_2}, u_{i_2+1}\} \cup \ldots \cup \{u_{i_n}, u_{i_n+1}\}, \quad i_{j+1} > i_j + 1$$

namely a disjoint union of pairs of consecutive members in U. $F = V' \cup F'$. Then F satisfies Gale's evenness criterion, it is a facet in $C(\nu, d+1)$. Call the collection of all facets in the form F as $\mathscr{E}$.

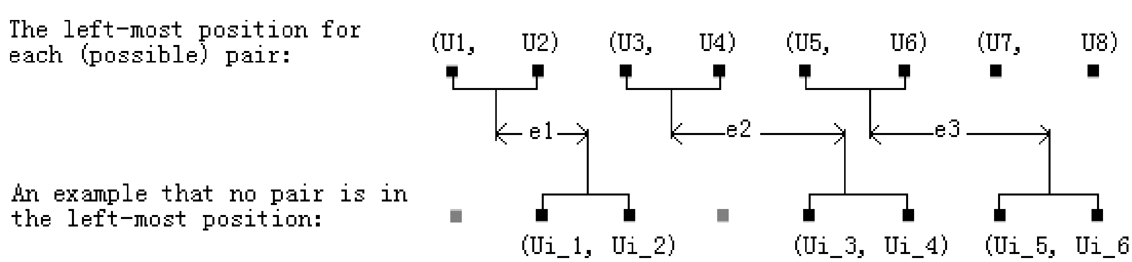
Let $\Phi^{(n)}$ denote the set of all monomials in g_1 variables with degree no more than n. We will build a bijection α between $\mathscr{E}$ and $\Phi^{(n)}$: for

$$F = V' \cup \{u_{i_1}, u_{i_1+1}\} \cup \{u_{i_2}, u_{i_2+1}\} \cup \ldots \cup \{u_{i_n}, u_{i_n+1}\},$$

define

$$\alpha(F) = Y_{e_1} Y_{e_2} \ldots Y_{e_n}, \quad e_j = i_j - 2j + 1, \quad 1 \le j \le n.$$

Intuitively e_j is the 'amount' by which the jth pair in F' is 'displaced' from its left-most position. We can see the series e_j are non decreasing,



so the map α is well defined. And whenever we have a monomial in $\Phi^{(n)}$ we can write it in the extended form, then it is easy to define the inverse map, α^{-1}. See the following example:

Example 3.3. Suppose $d = 8$, $n = 4$, $g_1 = 3$, the vertices on the moment curve are $\{v_1, u_1, u_2, \ldots, u_{11}\}$.
Take the monomial $Y_1Y_2Y_3$, which is $Y_0Y_1Y_2Y_3$ in the extended form, so

$$\alpha^{-1}(Y_0Y_1Y_2Y_3) = \{v_1\} \cup \{u_1, u_2\} \cup \{u_4, u_5\} \cup \{u_7, u_8\} \cup \{u_{10}, u_{11}\}.$$

Back to our definition, if we denote the total number of 'displaced' pairs in F as $\delta(F)$, we have $\delta(F) = \deg\ \alpha(F)$. Further, if we give both $\mathscr{E}$ and $\Phi^{(n)}$ the reverse lexicographic order, α is order preserving.

Another important fact is that whenever we have an M-sequence, there is an order ideal of monomials associated with it. We cite the following Lemma from [3] without proof.

Lemma 3.4. *Let $g = (g_0, g_1, g_2, \ldots)$ be a finite (or infinite) sequence of non negative integers. For each $i \geq 0$ let M_i be the first g_i monomials of degree i in the variables $Y_0, Y_1, Y_2, \ldots$. Define $M = \cup_{i \geq 0} M_i$. Then M is an order ideal of monomials iff g is an M-sequence.*

Now, consider the order ideal of monomials M constructed above. For a monomial m, if there is an $m' \in M$, such that $m < m'$ and deg m = deg m', we know $m \in M$, because elements of the same degree in M forms an initial set. When $m < m'$ and deg $m <$ deg m', we would easily find a devisor $\bar{m}$ of m' such that deg $\bar{m}$ = deg m and $m < \bar{m}$. Thus whenever $m <^* m'$ for some $m' \in M$ we would have $m \in M$.

If we write $\mathscr{B} = \alpha^{-1}(M)$ in the reverse lexicographic order $\{F_1, F_2 ..., F_\omega\}$, actually we would get a shelling order of the simplicial complex $\Delta = \cup_{i=1}^{\omega} \overline{F_i}$:

Lemma 3.5. *Δ is shellable with facets in reverse lexicographical order $F_1, F_2 ..., F_\omega$, and $s_k = \delta(F_k)$. Hence*

$$h_i(\Delta) = g_i,\ 0 \leq i \leq n; \quad 0,\ n+1 \leq i \leq d+1$$

Proof. Fixing k, $2 \leq k \leq \omega$, we will produce $\delta(F_k)$ elements $F_1^k, F_2^k ..., F_{\delta(F)}^k$ in $\mathscr{B}$, such that:
(i) they all come before F_k ;
(ii) $F_k \cap F_j^k$ are distinct sets of cardinality d ;
(iii) for all i, $i \leq k-1$, $F_k \cap F_i \subseteq F_k \cap F_j^k$ for some j ;

Denote $p = \delta(F_k)$. In F_k there are exactly p 'displaced' pairs:

$$F_k = V' \cup \{u_1, u_2, ..., u_{2n-2p}\} \cup \{u_{i_{n-p+1}}, u_{i_{n-p+1}+1}\} \cup \ldots \{u_{i_n}, u_{i_n+1}\},$$

where $i_{n-p+1} > 2n - 2p + 1$.

Put $G_j^k = F_k \backslash \{u_{i_j+1}\}$, $n-p+1 \leq j \leq n$. If $u_{j'}$ is the antecedent to the contiguous group containing u_{i_j+1}, we write

$$F_{j-n+p}^k = \{u_{j'}\} \cup G_{j-n+p}^k$$

then $F_i^k \in \mathscr{E}, 1 \leq i \leq p$; $F_k \cap F_j^k = G_j^k$ are p distinct sets of card d. As $F_j^k < F_k$ and $\delta(F_j^k) \leq \delta(F_k)$, obviously $\alpha(F_j^k) <^* \alpha(F_k)$. Recall M is an order ideal of monomials constructed in Lemma 3.4, we have $\alpha(F_j^k) \in M$. Thus $F_j^k \in \mathscr{B}$.
For any $F_i < F_k$ in $\mathscr{B}$, there is at least one $u_{i_j+1} \notin F_i \cap F_k$, $n-p+1 \leq j \leq n$. Then $F_i \cap F_k \subseteq F_{j-n+p}^k \cap F_k$.

Notice $s_k = \delta(F_k) =$ deg $\alpha(F_k)$. In our construction $|\{k : \text{deg } \alpha(F_k) = i\}| = g_i$. Thus from Lemma 3.5, $h_i(\Delta) = g_i,\ 0 \leq i \leq n; \quad 0,\ n+1 \leq i \leq d+1$. ☐

So we have built a simplicial complex whose h-vector is a given M-sequence g (and redundant zeroes). Now we show the boundary of Δ will provide us with a simplicial complex whose h-vector is the corresponding h-vector of g.

Observe in Δ any $(d-1)$-face can be at most in two different facets. We call Δ a pseudomanifold (see [8]). The boundary of a pseudomanifold is determined by the $(d-1)$-faces which are contained in only one facet. If the $(d-1)$-faces as $G_1, G_2, \ldots, G_\tau$, then the boundary $\partial\Delta = \cup_{i=1}^{\tau}\overline{G_i}$. For the set of faces not in the boundary, we denote them by Δ^o. As Δ is a shellable complex, by a well known result in [8], Δ is easily seen to be a topological d-ball. Thus $\partial\Delta$ is a $(d-1)$-sphere, and $h(\partial\Delta)$ satisfies Dehn-Sommerville equations. (without referencing to [8], we can still show $h(\partial\Delta)$ satisfies D-S equations, as $\partial\Delta$ is in fact polytopal. See end of this section.)

For the interior faces in Δ_k^o, we assert that they come from the interior faces of Δ_{k-1}^o, plus the faces in $\overline{F_k}$ which contain $\cap_{j=1}^{s_k}\overline{G_j^k}$. Indeed, when some boundary faces in Δ_{k-1}^o now turn into interior faces, they can only belong to $(d-1)$-faces in common of Δ_{k-1}^o and $\overline{F_k}$, so we need only care about the new interior faces in $\overline{F_k}$. As F_k is a d-simplex, the intersection of p distinct $(d-1)$-faces $\overline{G_j^k}$ is a $(d-p+1)$ set, name it A. If we want to recover A back to a $(d-1)$-face, we have to 'patch' $p-1$ points outside of it. It turns out we have at most $\begin{pmatrix} d+1-(d-p+1) \\ p-1 \end{pmatrix} = p$ choices, which are exactly the existing faces G_j^k in F_k. So any face containing A can not be in a $(d-1)$-face other than G_j^k. But G_j^k are exactly those $(d-1)$-faces shared by two facets in Δ, namely F_k and the F_j^k. So a face containing A cannot lie in the boundary. However, if a face do miss some points from A, then it must be included in a $(d-1)$-face other than G_j^k, thus it is in the boundary.

Under this criterion, we know newly-added interior faces in Δ_k will have dimension at least $d - s_k$. $f_j(\Delta_k^o) = f_j(\Delta_{k-1}^o)$, $-1 \leq j < d - s_k$. For $s = max\{s_k, 1 \leq k \leq \omega\}$, Δ has no interior face with dimension (strictly) smaller than $d - s$:

$$f_j(\Delta^o) = 0, \qquad f_j(\Delta) = f_j(\partial\Delta), \quad -1 \leq j < d - s.$$

Now if for some q, $h_i(\Delta) = 0$, $q < i \leq d+1$, which means $|\{k : s_k = i, i > q\}| = 0$, we would have the estimation $s \leq q$.

Now under our construction, as $h(\Delta) = (g_0, g_1, ..., g_n, 0, ..., 0)$, Δ has no interior face with dimension less than $d - n \geq n$:

$$(f_{-1}(\Delta), f_0(\Delta), ..., f_{n-1}(\Delta)) = f_{-1}(\partial\Delta), f_0(\partial\Delta), ..., f_{n-1}(\partial\Delta)).$$

By equation 2.2,

$$h(t) = (1-t)^{d+1} f(\frac{t}{1-t}) = \sum_{j=-1}^{d} f_j (1-t)^{d-j} t^{j+1}.$$

h_i is the coefficient of t^i, so it is only affected by f_j with $j < i$. Note that $\partial\Delta$ is one dimension less that Δ, so h_i is equal to the coefficient of t^i in $(1-t)h(\partial\Delta, t)$. Thus

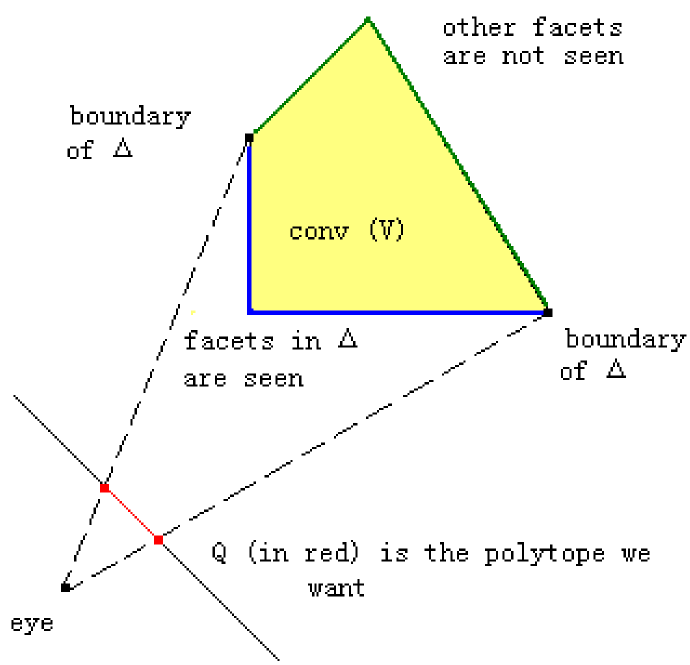
$$h_i(\Delta) = h_i(\partial\Delta) - h_{i-1}(\partial\Delta), \quad 1 \le i \le n.$$

On the other hand,

$$h_i(\Delta) = g_i = h_i - h_{i-1}, \quad 1 \le i \le n; \quad h_0 = h_0(\Delta) = h_0(\partial\Delta) = 1$$

we get $h_i(\partial\Delta) = h_i$, $0 \le i \le n$. Since $h(\partial\Delta)$ satisfies Dehn-Sommerville equations, h_i with $n < i \le d$ are symmetric with the first half, and $\partial\Delta$ is the simplicial complex with the given h-vector.

In [3] Billera and Lee further proved that $\partial\Delta$ is in fact polytopal, namely it is combinatorially equivalent to the boundary complex of a polytope. There is a theorem in [13] Section 5.2, which says whenever we have a polytope P and a point v outside of it (called an eye) in general position with all the facets in P , then the faces containing v in $conv(\{v\} \cup P)$ are exactly in the form $conv(\{v\} \cup F')$, where F'



belongs both to a facet of P that we can 'see' from v and a facet that we can not see. Billera and Lee allowed the points of the supporting set V to move on the moment curve (but keeps their order, thus keeps the combinatorics) and proved that in a proper position we can find a point v for the eye which 'sees' exactly the facets of $\mathscr{B}$. Then the faces in $\partial\Delta$ are reflected by the faces in $conv(\{v\} \cup V)$ which contains v. If then we 'cut' the neighborhood of v in $conv(\{v\} \cup V)$ with a codimension 1 hyperplane (called taking the vertex figure), we get a polytope combinatorially equivalent to $\partial\Delta$. See the above picture.

4. A Lower Bound for the Number of Simplicial Spheres

Now we follow Kalai to get a lower bound for the number of simplicial spheres. A *simplicial sphere* is a d-dimensional simplicial complex homeomorphic to a d-sphere (or namely a *triangulation* of a d-sphere.) A *PL sphere* is a simplicial sphere piecewise linear homeomorphic to the boundary of a simplex. And a PL sphere simplicially isomorphic to the boundary of a simplicial polytope is called a *polytopal sphere*. Let $c(d,n)$ be the number of polytopal $(d-1)$-spheres with n labelled vertices. In [12] Goodman and Pollack proved $\log c(d,n) \leq d(d-1)n \log n$. Kalai extended the construction methods we mentioned above and proved the number of triangulations of S^{d-1} with n labelled vertices $s(d,n)$ has a lower bound

$$\log s(d,n) \geq \frac{1}{(n-d)(d+1)} \binom{n - \lceil (d+2)/2 \rceil}{\lceil (d+1)/2 \rceil}.$$

Thus, when n or $n-d$ is big, only a small portion of simplicial spheres are polytopal. More surprisingly, for every $d \geq 5$ or $b \geq 4$,

$$\lim_{n\to\infty} \frac{c(d,n)}{s(d,n)} = 0, \quad \lim_{d\to\infty} \frac{c(d,d+b)}{s(d,d+b)} = 0.$$

Recall in the above section we find a shellable complex (the collection of facets in a cyclic polytope and that they correspond to an order ideal of monomials), and know it is a topologically ball. So its boundary forms a sphere. It seems we are very probable to attain a lot of simplicial spheres as long as we have many order ideal of monomials. In fact, in Kalai's modification we can have find many shellable simplicial balls (and thus simplicial spheres) with a more direct approach.

From now on we identify a facet in a cyclic polytope $C(n, d+1)$ with a d-set of integers $S = \{i_1, i_2, \ldots, i_d\}$, where $i_1 < i_2 < \ldots < i_d$ are the time t of the vertices on the moment curve. When the ball has at most n vertices, we assume i_j are taken from $[n] = \{1, 2, 3, \ldots, n\}$. Rather than reverse lexicographic order, now we adopt a *partial order*:

$S \leq T, \quad T = \{j_1, j_2, \ldots, j_d\}, \quad$ iff for every $1 \leq k \leq d, \quad i_k \leq j_k$.
We construct a facet:

$$F = \{i_1, i_1 + 1\} \cup \{i_2, i_2 + 1\} \cup \ldots \cup \{i_z, i_z + 1\},$$

with $z = (d + 1)/2$. Denote the set of all facets like this by $\mathscr{F}$. For simplicity we only consider the case when d is odd. (When d is even the argument is mostly the same.) Let $\mathscr{I}$ be a collection of facets. For a facet $F' \in \mathscr{I}$, if when $F \leq F'$ we have $F \in \mathscr{I}$, $\mathscr{I}$ is called an initial set (with respect to partial order). An initial set is also a shellable collection of facets, for we have the following Lemma:

Lemma 4.1. *Let $B(\mathscr{I})$ be the simplicial complex spanned by $\mathscr{I}$. Any order $F_1, F_2, \ldots, F_\omega$ of $\mathscr{I}$ compatible with the partial order is a shelling order of $B(\mathscr{I})$.*

The proof is quite similar to Lemma 3.5, we omit it here. We call $B(\mathscr{I})$ a *squeezed ball*, and its boundary a *squeezed sphere*. Luckily enough, we have as many simplicial spheres as simplicial balls:

Lemma 4.2. *Let B_1, B_2 be two squeezed balls. If $\partial B_1 = \partial B_2$ then $B_1 = B_2$.*

Proof. This Lemma actually means we can recover a squeezed ball from its boundary. Observe $B(\mathscr{I})$ is a pseudomanifold. In fact, a $(d-1)$-face $G \in B(\mathscr{I})$ has exactly one contiguous group (or end set) that is not even numbered, so we have at most two directions to add an integer to the odd numbered contiguous group (and only one direction to the end-set) to get a facet containing this face. Suppose $G \in F_1, F_2$, then it is not hard to see that the two facets are comparable. Assume $F_1 < F_2$. $\mathscr{I}$ is an initial set, so when $F_2 \in \mathscr{I}, \quad F_1 \in \mathscr{I}$, and G will not be in the boundary. So, when G is a $(d - 1)$-face in the boundary, F_1 must be a maximal facet in $\mathscr{I}$. Conversely, when F_1 is a maximal facet, there is at least one $(d - 1)$-face in $\partial B(\mathscr{I})$, e.g. the face missing the minimal integer in F_1. We know the boundary complex characterizes all the maximal facets in $\mathscr{I}$, thus the whole $\mathscr{I}$. □

The following Lemma provide further information about the squeezed sphere:

Lemma 4.3. *Let $S(\mathscr{I}) = \partial B(\mathscr{I})$ be a squeezed sphere. If i is a vertex of $S(\mathscr{I})$, then for every $j \leq i$, $j \in S(\mathscr{I})$.*

Proof. Since there is a facet in $\mathscr{I}$ that contains i, for every $j \leq i$ there is also a facet containing j. Choose a set which is maximal among all facets in $\mathscr{I}$ containing j, say F. Find in F a pair $\{i_k, i_k + 1\}$ which does not contain j, then $F \backslash \{i_k\} \in S(\mathscr{I})$. □

Now we can try to estimate how many squeezed spheres can we have. If we denote by $f(d,n)$ the number of squeezed $(d-1)$-spheres with *at most* n vertices, by Lemma 4.1 and 4.2 $f(d,n)$ is the number of initial sets in $\mathscr{F}$. By Lemma 4.3 the number of squeezed $(d-1)$-spheres with *exactly* n vertices is $f(d,n) - f(d,n-1)$.

So, all we need to know is the number of initial sets in $\mathscr{F}$. Initial sets are characterized by their maximal elements. However, in general it is not easy to presume how many 'maximal elements' can we find in an initial set (or equivalently, how many sets of 'incomparable elements' are there). Instead, observe in $\mathscr{I}$ that every maximal chain up to the same element has the same length. $\mathscr{I}$ is a ranked poset (in partial order). Of course any two elements with the same rank can not be comparable. If we write $B(n,d,r) = \{S \in \mathscr{F},\ rank(S) = r\}$, then we are sure any subset of $B(n,d,r)$ will generate a different initial set. Let

$$b(n,d,r) = |B(n,d,r)|, \quad b(n,d) = max\{b(n,d,r), r \geq 0\},$$

we have $f(n,d) \geq 2^{b(n,d)}$.

Suppose $F = \{i_1, i_1+1\} \cup \{i_2, i_2+1\} \cup \ldots \cup \{i_z, i_z+1\}$, we can obtain a maximal chain to F in the following elementary way. From left to right, in turn move the kth pair to their left-most position, e.g. if we move $\{i_k, i_k+1\}$ to $\{2k-1, 2k\}$, there are $i_k - 2k + 1$ positions in between. Now, denote $e_k = i_k - 2k + 1$,

$$rank(F) = \sum_{k=1}^{z} e_k.$$

Let $a(m,z,r)$ be the number of ways to represent r as the sum:

$$r = e_1 + e_2 + \ldots + e_z, \quad 0 \leq e_1 \leq e_2 \leq \ldots \leq e_z \leq m,$$

we have $b(n,d,r) = a(n-d+1, z, r)$. An easy argument shows

$$\sum_{0 \leq r \leq mz} a(m,z,r) = \binom{m+z}{z}.$$

For example we have m dots in a line, now use z boards to group them into $z+1$ segments (allow some segments to be empty), with e_k being the number of dots in the first k segments. It is equivalent to say there are totally $m+z$ positions in a line and choose any z of them to put the boards. Whatever the sum r is, it will always be in the range from 0 to mz.

Put $a(m,z) = max\{a(m,z,r),\ 0 \leq r \leq mz\}$, we have

$$a(m,z) \geq \frac{1}{1+mz}\binom{m+z}{z}.$$

$$b(n,d) \geq \frac{1}{1+(n-d-1)z}\binom{n-z}{z} \geq \frac{1}{(n-d)z}\binom{n-z}{z}.$$

Now we know $f(n,d) \geq 2^{b(n,d)}$, write $f(n,d) = 2^{b(n,d)} + g(n,d)$ where $g(n,d)$ is some positive function corresponding the number of initial sets not counted in $2^{b(n,d)}$. We can directly check $b(n,d)-1 \geq b(n-1,d)$. As any an initial set build on $[n-1]$ is also one that built on $[n]$, it is easy to see $g(n,d) \geq g(n-1,d)$. So

$$f(d,n) - f(d,n-1) \geq 2^{b(n,d)-1} + g(n,d) - g(n-1,d) \geq 2^{b(n,d)-1},$$

$$\log s(n,d) \geq \frac{1}{(n-d)z}\binom{n-z}{z}.$$

The lower bound is obtained.

Further, we say the h-vector of a squeezed sphere necessarily satisfies McMullen's conditions. We know immediately it satisfies the Dehn-Sommerville equations since it is a simplicial sphere([6]). Recall in Section 3 we have defined a map α from a collection of facets to monomials: $\alpha : \mathscr{F} \to \Phi^{(z)}$. When $\mathscr{I}$ is an initial set, it is not hard to find $\alpha(\mathscr{I})$ is an order ideal of monomials, e.g.:

Example 4.4. Suppose $d = 7$, $z = 4$, $Y_1Y_2Y_3$ is in $\alpha(\mathscr{I})$,

$$F_1 = \alpha^{-1}(Y_0Y_1Y_2Y_3) = \{1,2\} \cup \{4,5\} \cup \{7,8\} \cup \{10,11\}$$

then, if we look at Y_2Y_3:

$$F_2 = \alpha^{-1}(Y_0Y_0Y_2Y_3) = \{1,2\} \cup \{3,4\} \cup \{7,8\} \cup \{10,11\}$$

we have $F_2 < F_1$ in the partial order. Since $\mathscr{I}$ is an initial set, $F_2 \in \mathscr{I} \Rightarrow Y_2Y_3 \in \alpha(\mathscr{I})$.

When we have an order ideal of monomials M, define $k_i = |\{m \in M, \deg\ m = i\}|$, $i \geq 0$. We call $\{k_0, k_1, \ldots\}$ an *O-sequence*. Note in Lemma 3.4 we have a numerical characterization of the O-sequence of a (particularly constructed) order ideal of monomials : the O-sequence is in fact an M-sequence. While we do not need the full power of the following theorem to show the same characterization works for general OIM, as it is fundamental in the proof of Stanley for the necessity of McMullen's condition (for polytopes), we cite it here. Here we introduce some terms in commutative algebra, (see [1], [11], [17]). Let R be a Noetherian commutative ring with identity, graded by the nonnegative integers $\mathbb{N}$. The additive group of R has a direct sum decomposition:

$$R = R_0 + R_1 + R_2 + \ldots, \quad R_iR_j \subseteq R_{i+j},\ 1 \in R_0.$$

If R_0 is a field K we say R is a G-algebra. Since R is Noetherian it is finitely generated over the field K, and each R_i is a finite dimensional vector space over K. If further R is generated by elements in R_1, we call it a *standard G-algebra*. The Hilbert function of R is defined by $H(R, n) = \dim_K R_i,\ i \in \mathbb{N}$. In particular, $H(R, 0) = 1$. We have the following theorem:

Theorem 4.5. *(i) There exist a standard G-algebra R with $R_0 = K$ and Hilbert function H.*
(ii) $(H(0), H(1), H(2), \ldots)$ is an O-sequence.
(iii) $H(0) = 1$ and for $n \geq 1$, $H(n+1) \leq H(n)^{<n>}$.
(iv) Let $g = H(1)$ and M_n be the first (in reverse lexicographic order) $H(n)$ monomials of degree n in g variables. $M = \cup_{n \geq 0} M_n$ is an order ideal of monomials.

Proof. Note that $(iii) \Leftrightarrow (iv)$ is Lemma 3.4. We reference [17] for $(i) \Leftrightarrow (ii)$, and in [7] there is proof of $(ii) \Rightarrow (iv)$ involving only finite set arguments. $(iv) \Rightarrow (ii)$ is trivial. □

So, from $(ii) \Rightarrow (iv) \Rightarrow (iii)$ we know the O-sequence of an OIM is an M-sequence. Use the same arguments as in Section 3 we have $h_i(B(\mathscr{I})) = k_i$ for $0 \leq i \leq z$, where k_i is the ith component in the O-sequence of $\alpha(\mathscr{I})$, and $h_i = 0$ when $i > z$. (It is obvious that the highest degree in $\alpha(\mathscr{I})$ is z, since there are at most z pairs.) Similarly

$$\begin{aligned}&(1, h_1(S(\mathscr{I})) - h_0(S(\mathscr{I})), \ldots, h_z(S(\mathscr{I})) - h_{z-1}(S(\mathscr{I}))\ )\\&\quad = (1, h_1(B(\mathscr{I})), \ldots, h_z(B(\mathscr{I}))\ )\end{aligned}$$

is an M-sequence. We have proved the h-vector of $S(\mathscr{I})$ satisfies McMullen's conditions. However, we note it is still an open problem whether McMullen's conditions hold for *all* simplicial spheres.

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