AG homework 1

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1 On Spec(A)

1.Prove/disprove the following,

a) quasi compact. True.

b) $T \subseteq \mathcal{S}pec(A)$ is irreducible iff $\overline{T} = V(\mathfrak{p})$ for some $\mathfrak{p} \in \mathcal{S}pec(A)$.

Proof. True. T irreducible $\Rightarrow \overline{T}$ irreducible is obvious. For the other direction, if there exist two closed sets of $\mathcal{S}pec(A)$ not including each other, each has non trivial intersection with T and jointly cover T, then they would do so for $\overline{T}$. □

c) T is an irreducible component iff $\mathfrak{p} \in Specmin(A)$. True.

d) Every ideal α without coidempotents contains a minimal ideal α_0 without coidempotents, and α_0 consists of zero divisors only.

Proof. True. First each connected closed set is contained in a connected component $V(\alpha')$ where we can take α' to be a radical ideal. Since irreducible set is connected, $V(\alpha')$ contains some irreducible component $V(\mathfrak{p}_0)$ therefore $\alpha' \subseteq \mathfrak{p}_0$. Now we'll prove $\mathfrak{p}_0$ only has zero divisors. Since $\mathfrak{p}_0$ is minimal, localize and we'll see $\mathfrak{p}A_\mathfrak{p}$ is the nilradical of $A_\mathfrak{p}$ therefore every element is nilpotent in $A_\mathfrak{p}$ and a zero divisor in A.

Now we just need to show the existence of such a minimal (not necessarily radical) ideal. Define $\alpha_0 := \cap\{\beta | V(\beta) = V(\alpha')\}$. This is an ideal. We claim it doesn't have coidempotents. Suppose $f^2 - f = 0 \bmod \alpha_0$, then $f^2 - f = 0 \bmod$ each such β so $f = 0$ or $1 \bmod \beta$. However, since $\beta_1 + \beta_2 \subseteq \alpha'$ which is a radical ideal, f is either always 0 or always 1 mod different β, therefore $f = 0$ or $1 \bmod \alpha_0$. Obviously $V(\alpha_0)$ gives the same connected component, and α_0 is minimal. □

e) $T \subseteq \mathcal{S}pec(A)$ is connected iff $\overline{T} = V(\alpha)$ for some α without coidempotents.

Proof. "$\Rightarrow$" is obvious. "$\Leftarrow$" is false. Take two affine curve in $\mathbb{A}^2$ having an intersection pt, take T to be the union minus intersection pt, then $\overline{T}$ is connected while T is not. □

f) T is a connected component iff $T = V(\alpha_0)$ with α_0 a minimal ideal without coidempotents.

Proof. True. "$\Leftarrow$" by d). "$\Rightarrow$" we need to show if α_0 is a minimal ideal without coidempotents then $\sqrt{\alpha_0}$ is a minimal radical ideal without coidempotents. Suppose not, we have β a radical ideal without coidempotnts s.t. $\beta \subsetneq \sqrt{\alpha_0}$ then we have a new ideal without coidempotents $\beta \cap \alpha_0 \subsetneq \alpha_0$, because $\sqrt{\beta \cap \alpha_0} = \sqrt{\beta} \cap \sqrt{\alpha_0} \neq \sqrt{\alpha_0}$, contradiction. □

2. Prove/disprove:

a) $\mathfrak{p}$ is a closed point iff it is a maximal ideal. True.

b) Specmax(A) is a closed subspace of $\mathcal{Spec}(A)$ iff Specmax(A) is finite.

Proof. "$\Leftarrow$" is obvious. "$\Rightarrow$" is false. (It is true when A is Noetherian. Suppose Specmax=$V(\alpha)$, then A/α is a Noetherian affine scheme of dimension 0, thus necessarily has finite cardinality.) In general, there exists a zero dimension affine scheme with infinite cardinality, where every point is closed. By [H], every topological space which is T_0, quasi compact, has a quasi compact open basis which is stable under finite intersection, and where every non empty irreducible closed set has a generic point is called a spectral space. Such space can be realized as the prime spectral of some commutative ring. According to this, the p-adic integers Z_p is a spectral space. It is Hausdorff (thus every point is closed), so serves as a counter example. □

c) $\mathcal{Spec}(A)$ is Hausdorff iff Krulldim(A)=0.

Proof. "$\Rightarrow$" is obvious. "$\Leftarrow$" is Cor.2 Thm 1 [H] which says in a spectral space any two points either have disjoint neighborhood or belong to the closure of some (third) point. □

3. $\mathcal{Spec}(S^{-1}A) \to \mathcal{Spec}(A)$ is a homeomorphism onto its image. True.

2 Affine k-algebraic sets

4. Prove/disprove

a) $k^n = ((k^s)^n)^G$ and $(k^i)^n = ((\overline{k}^n)^G$. True.

b) V is a minimal affine k-algebraic set iff I(V) is a maximal ideal of $k[X_1 \dots X_n]$. True.

c) The minimal affine k-algebraic sets are exactly the G-orbits of $\overline{k}^n$.

Proof. Algebraic sets in K^n corresponds to radical ideals in $k[X_1 \dots X_n]$, and bigger ideals corresponding to vanishing set of smaller sets, therefore to find the minimal algebraic set $\Rightarrow$ maximal ideal $\Rightarrow$ vanishing of a point in K^n, or equivalently, kernel of the evaluation map at that point. We know the maximal ideals come from precisely evaluating at points in $\overline{k}^n$, by Hilbert Nullstellensatz. Points in $\overline{k}^n$ conjugated by G have the same evaluation kernel. Actually, when we are evaluating $k[X_1 \dots X_n]$ at point $\epsilon = (\epsilon_1 \dots \epsilon_n)$, X_i get identified with ϵ_i in the quotient field. But two algebraic extension over k which are isomorphic to each other could be moved around by an Galois action of G, therefore the points ϵ and ϵ' are conjugate. □

5. Finish the proof of Krull's Hauptidealsatz.

6. $n \geq 3$.

a) What are the irreducible components of $V_1 = V(2X_1^2 - 3X_2X_3)$, $V_2 = V(X_1X_2 - X_1)$, and $V_1 \cap V_2$?

V_1 irreducible. $V_2 = V(X_1) \cup V(X_2 - 1)$, $V_1 \cap V_2 = V(X_1, X_2) \cup V(X_1, X_3) \cup V(X_2 - 1, 2X_1^2 - 3X_3)$.

b) What are the connected components of the above sets?

They are all connected.

7. $X = \{(t, t^2, t^3) | t \in K\}$ is the twisted cubic curve, show it is an absolutely irreducible affine k-algebraic set of dimension 1.

Proof. Let's suppose n=3. $I(X) = (X_1^2 - X_2, X_1^3 - X_3)$, $\overline{k}[X_1, X_2, X_3]/I = \overline{k}[X_1]$ is a domain, therefore I is prime and X is absolutely irreducible. It is codimension 1. □

8. Prove/disprove:

a) State the corresponding statements when we replace K^n by V.

b) The closure of a point V_x is irreducible. Further, it is absolutely irreducible iff $x \in (k^i)^n$.

Proof. Any closed set containing x will contain everything, therefore V_x is irreducible. If $x \in (k^i)^n$ then V_x has only one point as a set in K^n. On the other hand, for any point $x \in \overline{(k)}^n \setminus (k^i)^n$ by c) we have finite but more than 1 points in its closure. Since $V_x \cap \overline{k}^n$ is nonempty and dense in V_x (see f) below), when we pass to $\overline{k}$, V_x would not be irreducible anymore. □

c) As x varies in K^n, $\dim(V_x)$ can be any number between 0 and n.

Proof. When $K = \overline{k}$, $\dim(V_x)$ can not be n, or in other words, V_x can not be K^n and $I(x)$ can not be the zero ideal. There always exist some polynomial f in the integral domain $k[X_1 \ldots X_n]$ which vanishes on the point x, and $k[X_1 \ldots X_n]/f$ will have dimension at most n-1. □

d) Describe the coordinate ring of the $k[V_x]$.

Proof. Since we are looking at $k[X_1 \ldots X_n]/I(x)$, we are looking at the image of the evaluation map at x, namely, we can replace every variable X_i by the i-th coordinate of x. □

e) $V \cap k^n$ is the set of Zariski closed points of V.

Proof. Actually $V \cap (k^i)^n$ is the set of closed points. Or, by 4. c) and a). □

f) $V \cap \overline{k}^n$ is Zariski dense in V.

Proof. True. Algebraic sets in K^n corresponds to radical ideals in $k[X_1 \dots X_n]$ which then corresponds to algebraic sets in $\overline{k}^n$, we conclude V and $V \cap \overline{k}^n$ corresponds to the same radical ideal $I \subset k[X_1 \dots X_n]$.

For any point x in V, suppose $D_{\overline{f}}$ is a neighborhood of x, for $\overline{f} \neq 0 \in k[V]$. Take a lift f, if $f(V \cap \overline{k}^n) = 0$ then by the Nullstellensatz applying to $V \cap \overline{k}^n$ we conclude $f \in I$ contradicts $f \neq 0 \in k[V]$. □

9. x is called a generic point of V if $V = \overline{x}$. Note however in algebraic set the generic point may not be unique. E.g, take $K = \mathbb{C}$, $k = \mathbb{Q}$, n=2, I is generated by $x^2 + y^2 - 1$, then $(\frac{2e}{e^2+1}, \frac{e^2-1}{e^2+1})$ is a generic point of V, if we replace e by π it is also a generic point.

a) If V has a generic point, then V is irreducible. See 8 b).

b) The converse is true iff $tr.deg(K|k) \geq dim(V)$.

Proof. True. Suppose V has a generic point $\epsilon \in K^n$, we'll show $tr.deg(K|k) \geq dim(V)$. Look at the function field $k(V)$ of $k[V]$, we want to embed $k(V)$ into K which will be achieved iff we can embed $k[V]$ into K. By definition, ϵ is a generic of V means if we replace the indeterminant x_i to the coordinate ϵ_i we get an isomorphism. Therefore we get this embedding.

For the other direction, (non canonically) let ϕ embed $k(V)$ (or its algebraic closure) into K, since two algebraically closed extension over a base field with the same transcendental degree are isomorphic. Take the generic point to have coordinate $(\phi(x_1) \dots \phi(x_n)$. In the example above, $\mathbb{Q}[V]$ is an integral extension of $\mathbb{Q}[x]$ and a transcendental degree 1 extension over $\mathbb{Q}$, take $\phi(x) = \frac{2e}{e^2+1}$, then $\phi(y) = \frac{e^2-1}{e^2+1}$ would allow us to embed k[V] into $\mathbb{C}$ which justifies that this point is indeed a generic point. □

c) The set of generic points of V is either empty or infinite.

Proof. False. E.g. take V to be a closed point, then we have a unique generic point. Actually, this set could be empty (violates condition in b)), finite (the point is in $\overline{k}^n$ and that there are finitely many of $k(V) \hookrightarrow K|k$) or infinite (k(V) is transcendental over k we can find infinitely many embeddings). □

References

[H] Hochster, Prime ideal structure of commutative rings, Trans of AMS, Vol.142, pp.43-60. *http://www.jstor.org/stable/1995344* 2