Units in a sextic extension

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Consider a cubic extension $\mathbb{K} := \mathbb{Q}(\alpha)$, when the minimal polynomial $f(x)$ of α does not totally split in $\mathbb{K}$. The normal closure $\mathbb{L} := \overline{(\mathbb{K})}$ is a sextic extension of $\mathbb{Q}$, with $Gal(\mathbb{L}/\mathbb{Q}) = \mathsf{S}_3$. Now we fix notation and pick one embedding of $\mathbb{K}$ as $\mathbb{K}_1$, say $\mathbb{K}_1$ is fixed by $(2,3) \in \mathsf{S}_3$. Here $(2,3)$ has the explicit description that if I choose one root u_1 of $f(x)$, $(2,3)$ permute the other two conjugate roots of u_1, since it has to act nontrivially on the three roots.

Notation: say the conjugate field K_2 is fixed by $(1,2)$, K_3 is fixed by $(1,3)$. $\mathbb{F}/\mathbb{Q}$ is the unique quadratic subextension inside $\mathbb{L}$, generated by the radical of the discriminant of $f(x)$. $\mathbb{F}$ is fixed by $(1,2,3)$.

Now we have two conditions,

- (1)$f(x)$ has three real roots and $\mathbb{L}$ is totally real. Consider the free Abelian multiplicative group of the units (mod torsion), we have $rk(\mathbb{U}_{\mathbb{L}}) = 5$, $rk(\mathbb{U}_{\mathbb{K}}) = 2$, $rk(\mathbb{U}_{\mathbb{F}}) = 1$.

- or (2)$f(x)$ has only one real root which we pick to be u_1 thus K_1 is thought as embedded in the reals. $\mathbb{L}$ has three pair of conjugate complex embeddings. $rk(U_{\mathbb{K}}) = 1$, $rk(U_{\mathbb{L}}) = 2$.

In both cases, the units in the conjugates $\mathbb{K}_i$ and $\mathbb{F}$ will generate a full rank sub lattice under the log map defined on $\mathbb{L}$, and we will discuss the index of this sub lattice.

1 The totally real case

$\mathbb{L}$ has six different real embeddings, call $\sigma_{1,0}, \sigma_{1,1}, \sigma_{2,0}, \sigma_{2,1}..\sigma_{3,1}$. Say $\sigma_{1,i}$ are extended above $\mathbb{K}_1$, $\sigma_{2,i}$ above $\mathbb{K}_2$, $\sigma_{3,i}$ above $\mathbb{K}_3$, $i = 0, 1$. Under our notation, (1,2,3) maps $\sigma_{1,i} \to \sigma_{2,i} \to \sigma_{3,i} \to \sigma_{1,i}$. Pick two fundamental units u_1, w_1 from $\mathbb{K}_1$. Their corresponding conjugates in K_i will be called u_i and w_i, for $i = 2, 3$. Denote $\tilde{u}_i := \log|u_i|$. $\tilde{u}_1 + \tilde{u}_2 + \tilde{u}_3 = 0$. $(1,2)$ will map $\tilde{u}_1$ to $\tilde{u}_2$ etc.

	Id$\to \sigma_{1,0}$	(1,2)$\to \sigma_{2,1}$	(1,3)$\to \sigma_{3,1}$	(2,3)$\to \sigma_{1,1}$	(1,2,3)$\to \sigma_{2,0}$	(1,3,2)$\to \sigma_{3,0}$
$h_{\tilde{u}_1}$	$\tilde{u}_1$	$\tilde{u}_2$	$\tilde{u}_3$	$\tilde{u}_2$	$\tilde{u}_3$	$\tilde{u}_1$

Here $\to$ means the action takes $\tilde{u}_1$ from the embedding of $\sigma_{1,0}$ and maps it to other embeddings. $h_{\tilde{u}_1}$ is a vector in $\mathbb{R}^6$ under the log map. If we write $\tilde{u}_3 = -\tilde{u}_1 - \tilde{u}_2$ and look at the 2-dim vector space generated by $\tilde{u}_1$ and $\tilde{u}_2$, we have a representation ρ:

$$\rho(1,2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \rho(1,3) = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \ \rho(2,3) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix},$$

$$\rho(1,2,3) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \ \rho(1,3,2) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$$

If we look at the character, this is in fact the unique irreducible 2-dim representation of S_3.

Now if we start with $\tilde{\mathsf{u}}_2$ and look at its image under the Galois action, we get another vector $h_{\tilde{\mathsf{u}}_2}$:

	Id→ $\sigma_{1,0}$	(1,2)→ $\sigma_{2,1}$	(1,3)→ $\sigma_{3,1}$	(2,3)→ $\sigma_{1,1}$	(1,2,3)→ $\sigma_{2,0}$	(1,3,2)→ $\sigma_{3,0}$
$h_{\tilde{\mathsf{u}}_2}$	$\tilde{\mathsf{u}}_2$	$\tilde{\mathsf{u}}_1$	$\tilde{\mathsf{u}}_2$	$\tilde{\mathsf{u}}_3$	$\tilde{\mathsf{u}}_3$	$\tilde{\mathsf{u}}_1$

In the same way we get 7 vectors under the log map: $h_{\tilde{\mathsf{u}}_i}$, $h_{\tilde{\mathsf{w}}_i}$, $i = 1..3$ and $h_{\tilde{\mathsf{v}}}$ where v comes from the fundamental unit of $\mathbb{F}$. In fact, v and its conjugate v' will give two vectors under the log map, but they are linearly dependent. By our assumption:

- u_1 and u_2 are in different subfields, so $\tilde{\mathsf{u}}_1$ and $\tilde{\mathsf{u}}_2$ are linearly indpendent over $\mathbb{Z}$, since u_1 is not a power of u_2 and vice versa.

- For each i, $\tilde{\mathsf{u}}_i$ and $\tilde{\mathsf{w}}_i$ are linearly independent over $\mathbb{Z}$ by assumption.

- $h_{\tilde{\mathsf{u}}_1}$ and $h_{\tilde{\mathsf{w}}_1}$ are linearly independent over $\mathbb{Q}$ as vectors. The determinant of one of their 2-minor will give us the regulator of $\mathbb{K}$. $\text{Reg}(\mathbb{K}) = |\tilde{\mathsf{u}}_1\tilde{\mathsf{w}}_2 - \tilde{\mathsf{u}}_2\tilde{\mathsf{w}}_1|$.

Now we show the 7 vectors h_i will generate a rank 5 lattice by showing $\tilde{\mathsf{u}}_1, \tilde{\mathsf{u}}_2, \tilde{\mathsf{w}}_1, \tilde{\mathsf{w}}_2, \tilde{\mathsf{v}}$ are linearly independent over $\mathbb{Z}$. Suppose

$$a_1\tilde{\mathsf{u}}_1 + a_2\tilde{\mathsf{u}}_2 + b_1\tilde{\mathsf{w}}_1 + b_2\tilde{\mathsf{w}}_2 = \tilde{\mathsf{v}}, \quad a_1, a_2, b_1, b_2 \in \mathbb{Z} \tag{1.1}$$

Let the group element (1,3) act on it, get

$$-a_1\tilde{\mathsf{u}}_1 + (a_2 - a_1)\tilde{\mathsf{u}}_2 - b_1\tilde{\mathsf{w}}_1 + (b_2 - b_1)\tilde{\mathsf{w}}_2 = -\tilde{\mathsf{v}} \tag{1.2}$$

Add (1.1) to (1.2), we have:

$$a_2\tilde{\mathsf{u}}_2 + b_2\tilde{\mathsf{w}}_2 = 0$$

thus $a_2 = b_2 = 0$. Replacing $\tilde{\mathsf{v}}$ by 0, we have $a_1 = b_1 = 0$. So $\tilde{\mathsf{u}}_1, \tilde{\mathsf{u}}_2, \tilde{\mathsf{w}}_1, \tilde{\mathsf{w}}_2, \tilde{\mathsf{v}}$ are independent over $\mathbb{Z}$. Another way to show their independence is to look at the regulator generated by

$$Mat_{5\times 6}(H) := \{h_{\tilde{\mathsf{u}}_1}, h_{\tilde{\mathsf{u}}_2}, h_{\tilde{\mathsf{w}}_1}, h_{\tilde{\mathsf{w}}_2}, h_{\tilde{\mathsf{v}}}\},$$

denoted by $Reg(V)$:

$$Reg(V) := Det(Minor_{5\times 5}(H)) = 9\tilde{\mathsf{v}}(\tilde{\mathsf{u}}_1\tilde{\mathsf{w}}_2 - \tilde{\mathsf{u}}_2\tilde{\mathsf{w}}_1)^2 = 9Reg(\mathbb{K})^2 Reg(\mathbb{F}).$$

Note in the expression of $Reg(V)$ the power of $Reg(\mathbb{K})$ and $Reg(\mathbb{F})$ show up exactly in the same way as they do in the Zeta function relation as we will show later. This is also the case when $\mathbb{L}$ is complex, though the constant 9 is replaced by 3. This implies we can compute the index of the subunits in $\mathbb{L}$ without computing any regulator of the sub fields. We will try to generalize this to fields $\mathbb{L}$ whose Galois group is a dihedral group D_{2p} where p is an odd prime. (Shafarevich showed every solvable group is realizable over $\mathbb{Q}$, which applies to the dihedral case in particular.)

2 The complex case

Now $\mathbb{K}$ has one real embedding, denoted by $\mathbb{K}_1$, and a pair of conjugate complex embeddings $\mathbb{K}_2$ and $\mathbb{K}_3$. $\mathbb{L}$ has three pair of complex embeddings. $rk(\mathbb{K}) = 1$, $rk(\mathbb{F}) = 0$, $rk(\mathbb{L}) = 2$. The units $\tilde{\mathsf{u}}_1$ and $\tilde{\mathsf{u}}_2$ generate a rank 2 lattice under the log map.

Note u_1 is real, u_2 and $u_3 = \overline{u}_2$ are complex, $\tilde{\mathsf{u}}_1 + 2\tilde{\mathsf{u}}_2 = \tilde{\mathsf{u}}_1 + \tilde{\mathsf{u}}_2 + \tilde{\mathsf{u}}_3 = 0$, $Reg(\mathbb{K}) = 2\tilde{\mathsf{u}}_2^2$. Note when computing the regulator in $\mathbb{L}$, for complex embeddings there is a constant 2 in front of $\tilde{\mathsf{u}}_i$:

$$Reg(V) = Det\begin{pmatrix} 2\tilde{\mathsf{u}}_1 & 2\tilde{\mathsf{u}}_2 \\ 2\tilde{\mathsf{u}}_2 & 2\tilde{\mathsf{u}}_3 \end{pmatrix} = 12\tilde{\mathsf{u}}_2^2 = 3Reg(\mathbb{K})^2$$

where $\tilde{\mathsf{u}}_1 = -2\tilde{\mathsf{u}}_2$.

3 Zeta function relations

Denote χ_1 as the trivial representation of S_3, χ_2 as the sign representation, χ_3 as the 2-dim representation. By working out the induced representations on the subgroups, we have:

$$\zeta(s,\mathbb{L}) = L(s,\chi_1)L(s,\chi_2)\mathbb{L}(s,\chi_3)^2$$

$$\zeta(s,\mathbb{K}) = L(s,\chi_1)L(s,\chi_3)$$

$$\zeta(s,\mathbb{F}) = L(s,\chi_1)L(s,\chi_2), \quad \zeta(s,\mathbb{Q}) = L(s,\chi_1)$$

Combining,

$$\zeta(s,\mathbb{Q})^2\zeta(s,\mathbb{L}) = \zeta(s,\mathbb{F})\zeta(s,\mathbb{K})^2 \tag{3.1}$$

Now for any global number field $\mathbb{K}$, we have the functional equation:

$$A^{1-s}\Gamma(\frac{1-s}{2})^{r_1}\Gamma^{r_2}(1-s)\zeta(1-s,\mathbb{K}) = A^s\Gamma(\frac{s}{2})^{r_1}\Gamma(s)^{r_2}\zeta(s,\mathbb{K}), \quad A := 2^{-r_2}d_{\mathbb{K}}^{\frac{1}{2}}\pi^{-\frac{r_1}{2}-r_2} \text{ is a constant.}$$

Here r_1 is the number of real embeddings, r_2 is number of pair of complex embeddings. $d_{\mathbb{K}}$ is the absolute value of the discriminant over $\mathbb{Q}$. This equation makes sense when all the function involved are analytic. When there is a pole or zero, we look at the limit, and use the $*$ notation. We will mainly be interested in the limit with $s \to 0$.

$$\lim_{s\to 0} sA^{1-s}\Gamma(\frac{1-s}{2})^{r_1}\Gamma^{r_2}(1-s)\zeta(1-s,\mathbb{K}) = \lim_{s\to 0} sA^s\Gamma(\frac{s}{2})^{r_1}\Gamma(s)^{r_2}\zeta(s,\mathbb{K}) \tag{3.2}$$

$$\lim_{s\to 0} -((1-s)-1)A^{1-s}\Gamma(\frac{1-s}{2})^{r_1}\Gamma^{r_2}(1-s)\zeta(1-s,\mathbb{K}) = \lim_{s\to 0} sA^s\Gamma(\frac{s}{2})^{r_1}\Gamma(s)^{r_2}\zeta(s,\mathbb{K}) \tag{3.3}$$

$$-A\pi^{\frac{r_1}{2}}\zeta^*(1,\mathbb{K}) = \Gamma^*(\frac{0}{2})^{r_1}\Gamma^*(0)^{r_2}\zeta^*(0,\mathbb{K}) \tag{3.4}$$

$$\zeta^*(1,\mathbb{K}) = -A^{-1}2^{r_1}\pi^{-\frac{r_1}{2}}\zeta^*(0,\mathbb{K}) \tag{3.5}$$

By the analytic class number formula, we have

$$\zeta^*(1,\mathbb{K}) = \frac{2^{r_1+r_2}\pi^{r_2}R_{\mathbb{K}}h_{\mathbb{K}}}{W_{\mathbb{K}}d_{\mathbb{K}}^{\frac{1}{2}}}, \tag{3.6}$$

$R_{\mathbb{K}}$ is the regulator, $W_{\mathbb{K}}$ is the cardinality of roots of unity.

Plugging in the formula, we have:

$$\frac{R_{\mathbb{K}}h_{\mathbb{K}}}{W_{\mathbb{K}}} = -\zeta^*(0,\mathbb{K}). \tag{3.7}$$

When $\mathbb{K} = \mathbb{Q}$, we have $\zeta^*(0,\mathbb{Q}) = -\frac{1}{2}$. By (3.1), we have:

$$\frac{R_{\mathbb{K}}^2R_{\mathbb{F}}}{R_{\mathbb{L}}} = \frac{1}{2^2}\frac{W_{\mathbb{K}}^2W_{\mathbb{F}}}{W_{\mathbb{L}}}\frac{h_{\mathbb{L}}}{h_{\mathbb{K}}^2h_{\mathbb{F}}}$$

In both the complex and real cases, $\mathbb{K}$ doesnot contain a third nor second roots of unity since it is a degree 3 extension. $\mathbb{L}$ contain i iff $\mathbb{F}$ does. So

$$\frac{1}{2^2}\frac{W_{\mathbb{K}}^2 W_{\mathbb{F}}}{W_{\mathbb{L}}} = 1 \Rightarrow \tag{3.8}$$

$$\frac{R_{\mathbb{K}}^2 R_{\mathbb{F}}}{R_{\mathbb{L}}} = \frac{h_{\mathbb{L}}}{h_{\mathbb{K}}^2 h_{\mathbb{F}}} \tag{3.9}$$

$$\frac{Reg(V)}{Reg_{\mathbb{L}}} = \frac{9h_{\mathbb{L}}}{h_{\mathbb{K}}^2 h_{\mathbb{F}}} \text{ when } \mathbb{L} \text{ is totally real} \tag{3.10}$$

$$\frac{Reg(V)}{Reg_{\mathbb{L}}} = \frac{3h_{\mathbb{L}}}{h_{\mathbb{K}}^2 h_{\mathbb{F}}} \text{ when } \mathbb{L} \text{ is imaginary} \tag{3.11}$$

$$\tag{3.12}$$

Note

- By comparing with the analytic formula without using the functional equation, we have

$$d_{\mathbb{L}} = d_{\mathbb{K}}^2 d_{\mathbb{F}}.$$

  Comparing with the tower relation for differents, this may show $\mathbb{K}$ and $\mathbb{F}$ are not disjoint in some cases, e.g. in the following Ishida cases. I would expect such a relation whenever there is a Bauer relation to exist bewteen the big field and its subfields. Is there a trivial way to get such a relation purely for discriminats without involving Norm operations as in the tower relation for differents?

- Consider the class of polynomial $f(x) = x^3 + lx - 1$ where is l is an even positive integer and $4l^3 + 27$ is square free. Such polynomial has a unique real root. One can show the index is 1, thus getting $3h_{\mathbb{L}} = h_{\mathbb{K}}^2 h_{\mathbb{F}}$.

4 Dihedral extensions

Consider a dihedral extension $\mathbb{L}/\mathbb{Q}$, with $Gal(\mathbb{L}/\mathbb{Q}) = D_{2p}$ where p is an odd prime. (How many such $\mathbb{L}$ can be totally real?) Give D_{2p} a presentation: $\{x, y | x^p = y^n = e,\ xy = yx^{-1}\}$. D_{2p} has a unique normal subgroup $\mathsf{X} := \{x^n,\ x = 1.., p-1\}$, whose fixed field we denote by $\mathbb{F}$. $\mathbb{F}$ is a quadratic extension of $\mathbb{Q}$. Pick another subgroup $\mathsf{Y} := \{e,\ y\}$ whose fixed field is denoted $\mathbb{K}$. All other degree p subfields $\mathbb{K}_i$ are conjugate to $\mathbb{K}$ since all elements $x^n y$ is conjugate to y (p is prime). Since $\mathsf{S}_3 = D_3$, this would include the previous consideration as a special case.

Now we work out the irreducible representations of D_{2p}. We have the trivial representation χ_1, the 'sign' representation χ_2 with kernel X, and other $\frac{p-1}{2}$ distinct irreducible 2-dim representations are induced by the cyclic characters on X: $\rho^i_{\mathsf{X}}(x) = \omega^i$ where ω is a primitive pth root of unity. We abuse notation and denote the induced representation of ρ^i_{X} to D_{2p} as ρ^i, here i is an index not a power. It is elementary to show the ρ^i are irreducible, and $tr(\rho^i(x)) = 2\cos(\frac{2i}{p})$. Pick $i = 1, 2, ...\frac{p-1}{2}$ their trace are all distinct, and

$$1 + 1 + \sum_{i=1,2,...\frac{p-1}{2}} dim^2_{\rho^i} = 1 + 1 + \frac{p-1}{2} \times 4 = 2p.$$

Thus we have all the irreducible representations.

We have decomposition of the zeta function

$$\zeta(s, \mathbb{L}) = L(s, \chi_1)L(s, \chi_2)\Pi_i L(s, \rho^i)^2$$

Each ρ^i restrict to Y has fixed dimension 1, since if we pick $\mathsf{X} \otimes v$, $y\mathsf{X} \otimes v$ as basis of the induced representation,

$$\rho^i(y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

χ_2 restricts to Y do not have fixed subspace. So

$$\zeta(s,\mathbb{K}) = L(s,\chi_1)\Pi_i L(s,\rho^i)$$

In similar way we get

$$\zeta(s,\mathbb{F}) = L(s,\chi_1)L(s,\chi_2)$$

Combining,

$$\zeta^2(s,\mathbb{Q})\zeta(s,\mathbb{L}) = \zeta^2(s,\mathbb{K})\zeta(s,\mathbb{F}).$$