8. Find the average value of the function $f(x) = x\cos 2x$ on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. (10 points)

$$f_{ave} = \frac{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)\,dx}{\frac{\pi}{2} - (-\frac{\pi}{2})}$$

$$= \frac{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos 2x\,dx}{\pi}$$

Let's find $\int x\cos 2x\,dx$

Let $u = 2x$ $\quad x = \frac{u}{2}$, $\quad dx = \frac{du}{2}$

so $\int x\cos 2x\,dx = \int \frac{u}{2}\cos u\,\frac{du}{2}$

$$= \frac{1}{4}\int u\cos u\,du$$

Let $t = 2x$ $\quad x = \frac{t}{2}$, $\quad dx = \frac{dt}{2}$

so $\int x\cos 2x\,dx = \int \frac{t}{2}\cos t\,\frac{dt}{2} = \frac{1}{4}\int t\cos t\,dt$

Then Integration by parts:

$u = t \quad u' = 1$

$v = \sin t \quad v' = \cos t$

$$\frac{1}{4}\int t\cos t\,dt = \frac{1}{4}\left(t\sin t - \int \sin t\,dt\right)$$

$$= \frac{1}{4}\left(t\sin t + \cos t + C\right)$$

$$= \frac{1}{4}\left(2x\sin 2x + \cos 2x + C\right)$$