Riemannian Geometry

1.2.1 Definition

A Riemannian metric on a C^∞ manifold M is a correspondence, which associates to each point $p \in M$, an inner product $(-,-)_p$ (a symmetric bilinear positive-definite form), on the tangent space T_pM, which varies differentiably.

1.2.10 Proposition

A differentiable manifold M has a Riemannian metric.

2.2.1 Definition

An affine connection ∇ on a C^∞ mfd M is a map
($\mathfrak{X}(M)$ is the space of vector fields on M)

$$\nabla: \ \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

(denoted by $(X,Y) \to \nabla_X Y$), s.t.
(1) $\nabla_{fX+gY} Z = f \cdot \nabla_X Z + g \cdot \nabla_Y Z$,
(2) $\nabla_X (Y+Z) = \nabla_X Y + \nabla_X Z$,
(3) $\nabla_X (fY) = f \nabla_X Y + X(f) \cdot Y$,
for any $X, Y, Z \in \mathfrak{X}(M)$, and $f, g \in C^\infty(M)$.

2.2.2 Proposition

M is a smooth mfd with an affine connection ∇. Then $\exists!$ correspondence which associates to a vector field V along the smooth curve $c: I \to M$, another vector field $\frac{DV}{dt}$ along c, called the covariant derivative of V along c, such that:
(1) $\frac{D}{dt}(V+W) = \frac{DV}{dt} + \frac{DW}{dt}$,
(2) $\frac{D}{dt}(fV) = \frac{df}{dt} \cdot V + f \cdot \frac{DV}{dt}$,
(3) if V is induced by a vector field $Y \in \mathfrak{X}(M)$, then $\frac{DV}{dt} = \nabla_{\frac{dc}{dt}} Y$.

2.2.3 Remark

The vector $(\nabla_X Y)_p$ depends only on the value of X at p, and the value of Y along a curve tangent to $X(p)$.

2.2.6 Proposition

We can parallel translate (s.t. $\frac{D}{dt} = 0$) a vector along a curve.

2.3.1 Definition

M is a smooth mfd with a Riemannian metric, and an affine connection, then the connection is compatible with the metric if for any curve c and any pair of parallel vector fields P and P' along c, we have (P, P') is constant.

2.3.1 Proposition

∇ is compatible with $(-,-)$ iff for any vector fields V, W along any curve c,

$$\frac{d}{dt}(V, W) = (\frac{DV}{dt}, W) + (V, \frac{DW}{dt}).$$

2.3.3 Corollary

∇ is compatible with $(-,-)$ iff for any $X, Y, Z \in \mathfrak{X}(M)$,

$$X(Y, Z) = (\nabla_X Y, Z) + (Y, \nabla_X Z).$$

2.3.4 Definition

∇ is symmetric if for any $X, Y \in \mathfrak{X}(M)$,

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

2.3.6 Theorem (**Levi-Civita**)

Given a Riemannian mfd M, $\exists!$ affine connection ∇ s.t.

(1) ∇ is symmetric,

(2) ∇ is compatible with the metric.

It is called the Levi-Citita connection or Riemannian connection.

It is given by

$$\begin{aligned}(Z, \nabla_Y X) &= \frac{1}{2}\Big\{X(Y, Z) + Y(Z, X) - Z(X, Y) \\ &\quad -([X, Z], Y) - ([Y, Z], X) - ([X, Y], Z)\Big\}.\end{aligned}$$

2.3.- (Computation)

Fix a coordinate chart (U, x). Let Γ_{ij}^k (the Christoffel symbols) be determined by

$$\nabla_{X_i} X_j = \sum_k \Gamma_{ij}^k X_k.$$

Then if we let $g_{ij} = (X_i, X_j)$, and the matrix (g^{km}) be the inverse of (g_{km}), they can be expressed by

$$\Gamma_{ij}^m = \frac{1}{2}\sum_k \left\{ \frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ki} - \frac{\partial}{\partial x_k} g_{ij} \right\} g^{km}.$$

And,

$$\frac{DV}{dt} = \sum_k \left\{ \frac{dv^k}{dt} + \sum_{i,j} \Gamma_{ij}^k v^j \frac{dx_i}{dt} \right\} X_k$$

3.2.1 Definition

Curve $\gamma : I \to M$ is a geodesic at $t_0 \in I$ if $\frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = 0$ at t_0. If γ is a geodesic at all points, then it is a geodesic. If $|\frac{d\gamma}{dt}| = 1$, then γ is normalized.

3.2.- (Computation)

Geodesic should satisfy: for each k,

$$\frac{d^2 x_k}{dt^2} + \sum_{i,j} \Gamma_{ij}^k \frac{dx_i}{dt}\frac{dx_j}{dt} = 0.$$

3.2.2 Theorem

Any smooth vector field has a unique flow (locally).

3.2.3 Lemma

There is a unique vector field G on TM whose trajectories are of the form $t \to (\gamma(t), \gamma'(t))$, where γ is a geodesic on M.

3.2.4 Definition

The vector field G is called the geodesic field, and its flow is called the geodesic flow.

3.2.9 Proposition

Given $q \in M$, there is an $\epsilon > 0$, s.t. $\exp_q : B_\epsilon(0) \to M$ is a diffeomorphism onto an open subset of M.

3.3.5 **Gauss Lemma**

Let $p \in M$ and $v \in T_pM$ with $\exp_p v$ is defined. Let $w \in T_pM \cong T_v(T_pM)$. Then $\big((d\exp_p)_v(v), (d\exp_p)_v(w)\big) = (v, w)$.

3.3.6 Proposition

In a geodesic ball, geodesic from the center is minimizing.

3.4.2 Proposition

For any $p \in M$, there is a number $\beta > 0$ such that $B_\beta(p)$ is strongly convex (for any two points in its closure, there is unique minimizing geodesic joining them whose interior is in it).

3.- (Exercises)

A Killing field $X \in \mathfrak{X}(M)$, is a vector field whose flow preserves the metric tensor.

X is Killing $\Leftrightarrow (\nabla_Y X, Z) + (\nabla_Z X, Y) = 0$ for all $Y, Z \in \mathfrak{X}(M)$.

A Killing field X is uniquely determined by X_p and $(\nabla X)_p$.

3.- **Gradient, Divergence**

$f : M \to \mathbb{R}$. Then define the vector field gradient of f by

$$(\mathrm{grad} f(p), v) = df_p(v), \quad p \in M, \ v \in T_pM.$$

X is a vector field on M. Define the function divergence of X by

$$\mathrm{div} X(p) = \mathrm{tr}(\nabla_{(-)} X : T_pM \to T_pM).$$

If we let ω be the volume form near p, then

$$\mathrm{div} X(p) \cdot \omega = L_X \omega.$$

In analogy, define the divergence of a $(1, r)$-tensor S to be the $(0, r)$-tensor

$$(\mathrm{div} S)(v_1, \ldots, v_r) = \mathrm{tr}(w \to (\nabla_w S)(v_1, \ldots, v_r)).$$

3.- **Laplacian**

Define the Laplacian operator $\triangle : C^\infty(M) \to C^\infty(M)$ by

$$\triangle f = \mathrm{div}(\mathrm{grad} f).$$

4.2.1 Definition

The <u>curvature</u> of a Riemannian mfd M associates to every pair $X, Y \in \mathfrak{X}(M)$ a map, $R(X,Y) : \mathfrak{X}(M) \to \mathfrak{X}(M)$, given by

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z.$$

(Other authors may define in the opposite sign.)

4.2.2 Proposition

Curvature $R(-,-)$ satisfies:
(1) $C^\infty(M)$-bilinear in $\mathfrak{X}(M) \times \mathfrak{X}(M)$.
(2) $R(X,Y) : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is $C^\infty(M)$-linear.

4.2.4 Proposition (Bianchi Identity)

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0.$$

4.2.- (Convention)

We write $(R(X,Y)Z, T) = (X, Y, Z, T)$.

4.2.5 Proposition
(1) $(X,Y,Z,T) + (Y,Z,X,T) + (Z,X,Y,T) = 0$,
(2) $(X,Y,Z,T) = -(Y,X,Z,T)$,
(3) $(X,Y,Z,T) = -(X,Y,T,Z)$,
(4) $(X,Y,Z,T) = (Z,T,X,Y)$.

4.2.- (Notation)

Let R^l_{ijk} be determined by

$$R(X_i, X_j)X_k = \sum_l R^l_{ijk} X_l.$$

Then by direct calculation

$$R^s_{ijk} = \sum_l \Gamma^l_{ik} \Gamma^s_{jl} - \sum_l \Gamma^l_{jk} \Gamma^s_{il} + \frac{\partial}{\partial x_j} \Gamma^s_{ik} - \frac{\partial}{\partial x_i} \Gamma^s_{jk}.$$

We also let $(X_i, X_j, X_k, X_l) = R_{ijkl}$.

4.3.2 Proposition

Let σ be a 2-dim subspace of T_pM, and x, y span it. Then the <u>sectional curvature</u> of σ

$$K(\sigma) = K(x, y) = \frac{(x, y, x, y)}{|x \wedge y|^2}$$

does not depend on the choice of x, y.

4.3.3 Lemma

Sectional curvatures uniquely determine curvature.

4.3.4 Lemma

M has constant sectional curvature equal to K_0 if and only if: $R = K_0 \cdot R'$ where $R' : T_pM \times T_pM \times T_pM \to T_pM$ is determined by

$$(R'(X, Y)W, Z) = (X, W)(Y, Z) - (Y, W)(X, Z),$$

or equivalently,

$$R(X, Y)W = K_0 \cdot [(X, W)Y - (Y, W)X].$$

4.3.5 Corollary

$K(\sigma) = K_0$ for all $\sigma \subset T_pM$ if and only if: $R_{ijij} = -R_{ijji} = K_0$ for $i \neq j$ and $R_{ijkl} = 0$ otherwise (under orthonormal basis at p).

4.4.- (Definition)

<u>Ricci tensor</u>, Ric : $T_pM \times T_pM \to \mathbb{R}$, is given by: for any $u, v \in T_pM$, $\text{Ric}(u, v)$ is $\frac{1}{n-1}$ times the trace of $R(u, -)v : T_pM \to T_pM$.

If x is a unit vector, we let $\text{Ric}(x) = \text{Ric}(x, x)$ be the <u>Ricci curvature</u> in the direction x.

Let $\{x, z_1, z_2, \ldots, z_{n-1}\}$ be an orthonormal basis of T_pM, then

$$\text{Ric}(x) = \frac{1}{n-1} \sum_i (R(x, z_i)x, z_i) \, .$$

<u>Scalar curvature</u> at p, is defined to be (here $\{x_1, \ldots, x_n\}$ is a basis for T_pM)

$$K(p) = \frac{1}{n} \sum_j \text{Ric}(x_j) = \frac{1}{n(n-1)} \sum_{i,j} (R(x_i, x_j)x_i, x_j).$$

Let bilinear form $Q(-, -)$ on T_pM be defined as: $Q(u, v)$ is trace of $R(u, -)v : T_pM \to T_pM$. Then $Q(-, -) = (n-1) \cdot \text{Ric}(-, -)$. This Q is called Ricci tensor

for lots of authors.
The symmetric bilinear form Q corresponds to a linear self-adjoint map $K : T_pM \to T_pM$, given by $(K(-), -) = Q(-, -)$. Then let $\{z_1, \ldots, z_n\}$ be an orthonormal basis, we have

$$\mathrm{tr}K = \sum_j (Kz_j, z_j) = \sum_j Q(z_j, z_j) = (n-1) \sum_j \mathrm{Ric}_p(z_j) = n(n-1)K(p).$$

So $n(n-1)$ times the scalar curvature $K(p)$ is the trace of $Q(-, -)$. Therefore, Ricci curvature and scalar curvature are independent of the basis chosen.

4.4.- (Computation)

Let $R_{ik} = Q(X_i, X_k)$, then

$$\frac{1}{n-1} Q(X_i, X_k) = \frac{1}{n-1} R_{ik} = \frac{1}{n-1} \sum_j R^j_{ijk} = \frac{1}{n-1} \sum_{s,j} R_{ijks} g^{sj}.$$

And

$$K = \frac{1}{n(n-1)} \sum_{i,k} R_{ik} g^{ik}.$$

4.4.1 Lemma

Let $f : A(\subset \mathbb{R}^2) \to M$ be a parameterized surface, and (s, t) be the usual coordinates of $\mathbb{R}^2$. Let V be a vector field along f.

$$\frac{D}{\partial t}\frac{D}{\partial s} V - \frac{D}{\partial s}\frac{D}{\partial t} V = R(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}) V$$

4.5.1 Definition

A tensor T of order r on a Riemannian manifold is a C^∞-multilinear map (from $\mathfrak{X}(M)^r$)

$$T : \ \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C^\infty(M).$$

4.5.- (Examples)

Curvature tensor $(R(-, -)-, -)$ of order 4.
Metric tensor $(-, -)$ of order 2.
Connection ∇ or $(\nabla_X Y, Z)$ is not a tensor, because it is not C^∞-linear on Y.

4.5.7 Definition

Let T be a tensor of order r. The covariant differential ∇T of T is a tensor of order $(r+1)$ given by

$$\begin{aligned}\nabla T(Y_1,\dots,Y_r,Z) &= Z\left(T(Y_1,\dots,Y_r)\right)\\ &\quad -T(\nabla_Z Y_1,\dots,Y_r)-\cdots-T(Y_1,\dots,Y_{r-1},\nabla_Z Y_r).\end{aligned}$$

For each $Z\in\mathfrak{X}(M)$, the covariant derivative $\nabla_Z T$ of T relative to Z, is a tensor of order r given by

$$\nabla_Z T(Y_1,\dots,Y_r)=\nabla T(Y_1,\dots,Y_r,Z).$$

4.- (Exercises)

Let$G$ be a Lie group with a bi-invariant metric. Let X,Y,Z be left-invariant vector fields with unit length. Then:

(a) $\nabla_X Y=\frac{1}{2}[X,Y]$;

(b) $R(X,Y)Z=\frac{1}{4}\big[[X,Y],Z\big]$;

(c) If $X\perp Y$, and σ is the plane of X,Y, then the sectional curvature

$$K(\sigma)=\frac{1}{4}\parallel[X,Y]\parallel^2 .$$

Therefore, the sectional curvature $K(\sigma)$ of a Lie group with bi-invariant metric is non-negative, and is zero if and only if $[X,Y]=0$.

Riemannian manifold M is called locally symmetric is $\nabla R=0$

Second Bianchi Identity

$$(\nabla_u R)(v,w)+(\nabla_v R)(w,u)+(\nabla_w R)(u,v)=0$$

Schur's Theorem

M^n connected, $n\geq 3$. M is isotropic, which means $\forall p\in M$ the sectional curvature $K(p,\sigma)$ does not depend on $\sigma\subset T_pM$. Then it does not depend on p either.

5.2.1 **Jacobi Field**

Let $\gamma : [0,a] \to M$ be a geodesic in M. A vector field J along γ is a Jacobi field if it satisfies the Jacobi equation

$$\left.\frac{D^2 J}{dt^2}\right|_t + R(\gamma'(t), J(t))\gamma'(t) = 0$$

for all $t \in [0,a]$.

5.2.7 Proposition

Let $\gamma : [0,a] \to M$ be a geodesic with $\gamma(0) = p$, $\gamma'(0) = v$. Let $w \in T_v(T_pM)$ with $|w| = 1$. Let J be a Jacobi field along γ given by

$$J(t) = (d\exp_p)_{tv}(tw), \quad 0 \le t \le a.$$

Then the Taylor expansion of $|J(t)|^2$ about $t = 0$ is

$$|J(t)|^2 = t^2 - \frac{1}{3}(R(v,w)v, w) \cdot t^4 + o(t^4).$$

5.2.10 Corollary

If in addition $|v| = 1$, and $v \perp w$, then

$$|J(t)| = t - \frac{1}{6}K(p,\sigma)t^3 + o(t^3)$$

where σ is the plane of v, w.

5.3.1 Definition

Let $\gamma : [0,a] \to M$ be a geodesic. The point $\gamma(t_0)$ is conjugate to $\gamma(0)$, if $\exists$ a Jacobi field J along γ, not identically zero, with $J(0) = 0 = J(t_0)$.

5.3.5 Proposition

Conjugate points of p, are precisely image of critical points of $\exp_p$.

6.2.1 **Second Fundamental Form**

$U \subset M$ is a submanifold. The map $B : \mathfrak{X}(U) \times \mathfrak{X}(U) \to \mathfrak{X}(U)^\perp$ given by

$$B(X,Y) = \overline{\nabla}_{\overline{X}}\overline{Y} - \nabla_X Y$$

is bilinear and symmetric.

6.2.-

$M \subset \overline{M}$ submanifold. $p \in M$ and $\mu \in (T_pM)^\perp \subset T_p\overline{M}$.

Define map $H_\mu : T_pM \times T_pM \to \mathbb{R}$ to be such that

$$H_\mu(x, y) = (B(x, y), \mu).$$

Then H_μ is a symmetric bilinear form.

6.2.2 Definition

The quadratic form $\mathbb{I}_\mu$ defined on T_pM by

$$\mathbb{I}_\mu(x) = H_\mu(x, x)$$

is called the second fundamental form of f (the immersion) at p along the normal vector μ.

6.2.- **Shape Operator**

Let Shape operator $S_\mu : T_pM \to T_pM$ be defined by

$$(S_\mu(x), y) = H_\mu(x, y) = (B(x, y), \mu).$$

Then S_μ is self-adjoint.

6.2.3 Proposition

Let $x \in T_pM$, and $\mu \in (T_pM)^\perp$. Let N be a local extension of μ normal to M. Then

$$S_\mu(x) = -(\overline{\nabla}_x N)^T.$$

6.2.- (Definition)

Consider immersion $f : M^n \to \overline{M}^{n+1}$. Let $\mu \in (T_pM)^\perp$, $|\mu| = 1$. Since S_μ is symmetric, $\exists$ eigenvectors $\{e_1, \ldots, e_n\}$ from an orthonormal basis, with eigenvalues $\lambda_1, \ldots, \lambda_n$. We say the e_i are principal directions, and λ_i are principal curvatures of f.

$\det(S_\mu) = \lambda_1 \cdots \lambda_n$ is called the Gauss-Kronecker curvature of f.

$\frac{1}{n}(\lambda_1 + \cdots + \lambda_n)$ is called the mean curvature.

If $\overline{M} = \mathbb{R}^{n+1}$, let N be a unit vector field (locally defined) normal to M, and define the Gauss map $g : M^n \to S^n$ by

$$g(q) = N(q).$$

Then $dg_q(x) = \frac{d}{dt}N(c(t))\big|_{t=0} = \overline{\nabla}_x N = (\overline{\nabla}_x N)^T = -S_\mu(x)$.

6.2.5 Theorem (Gauss)

$p \in M \in \overline{M}$, and x, y are orthonormal in T_pM, then

$$K(x,y) - \overline{K}(x,y) = \big(B(x,x), B(y,y)\big) - |B(x,y)|^2.$$

6.2.- (definition)

Immersion $f : M \to \overline{M}$ is geodesic at $p \in M$ if every second fundamental form is zero at p. Also say f is totally geodesic if it is geodesic for all $p \in M$.

6.2.10 Definition

Immersion $f : M \to \overline{M}$ is minimal if, for every $p \in M$ and every $\mu \in (T_pM)^\perp$, the trace of S_μ is 0.

6.3.- Normal connection

$f : M^n \to \overline{M}^{n+m}$ isometric immersion. Then $T_p\overline{M} = T_pM \oplus (T_pM)^\perp$, the direct sum of tangent bundle and normal bundle. Given X a tangent vector field, and η a normal vector field, the tangent component of $\overline{\nabla}_X\eta$ is

$$(\overline{\nabla}_X\eta)^T = -S_\eta(X).$$

The normal component of $\overline{\nabla}_X\eta$ is

$$\begin{aligned} \nabla^\perp_X\eta &= (\overline{\nabla}_X\eta)^\perp = \overline{\nabla}_X\eta - (\overline{\nabla}_X\eta)^T \\ &= \overline{\nabla}_X\eta + S_\eta(X). \end{aligned}$$

(Above is called Weingarten equation.) This normal connection $\nabla^\perp$ has all of the usual properties of a connection (C^∞-linear in X, $\mathbb{R}$-linear in η, and Leibniz rule).

The normal curvature $R^\perp$ is defined by

$$R^\perp(X,Y)\eta = \nabla^\perp_Y\nabla^\perp_X\eta - \nabla^\perp_X\nabla^\perp_Y\eta + \nabla^\perp_{[X,Y]}\eta.$$

6.3.1 **Gauss equation, Ricci equation**

(a)

$$(\overline{R}(X,Y)Z,T) = (R(X,Y)Z,T) - (B(Y,T),B(X,Z)) + (B(X,T),B(Y,Z)).$$

(b)

$$(\overline{R}(X,Y)\eta,\zeta) - (R^{\perp}(X,Y)\eta,\zeta) = ([S_\eta, S_\zeta]X, Y),$$

where $[S_\eta, S_\zeta] = S_\eta S_\zeta - S_\zeta S_\zeta$.

6.3.4 Codazzi's equation

$$(\overline{R}(X,Y)Z,\eta) = (\overline{\nabla}_Y B)(X,Z,\eta) - (\overline{\nabla}_X B)(Y,Z,\eta),$$

where

$$\begin{aligned} B(X,Y,\eta) &= (B(X,Y),\eta), \\ (\overline{\nabla}_X B)(Y,Z,\eta) &= X(B(Y,Z,\eta)) - B(\nabla_X Y,Z,\eta) - B(Y,\nabla_X Z,\eta) - B(Y,Z,\nabla^{\perp}_X \eta). \end{aligned}$$

If the codimension is 1, then it implies

$$\nabla_X(S_\eta(Y)) - \nabla_Y(S_\eta(X)) = S_\eta([X,Y]).$$

7.2.8 **Rinow-Hopf**

$p \in M$. The following are equivalent:

(a) $\exp_p$ id defined for all of T_pM;

(b) Closed bounded sets of M are compact;

(c) M is complete as a metric space;

(d) M is geodesically complete;

(e) There is a sequence of compact subsets of M, $K_1 \subset K_2 \subset \cdots$, and $\cup_n K_n = M$, such that if $q_n \notin K_n$ then $d(p,q_n) \to \infty$.

In addition, any of the above implies

(f) For any $q \in M$ there is a minimizing geodesic joining p, q.

7.2.9 Corollary

If M is compact, then it is complete.

7.3.1 **Hadamard Theorem**

M is complete, simply connected, with all sectional curvatures non-positive. Then for any $p \in M$, the map $\exp_p$ is a diffeomorphism from T_pM to M.

8.2.1 **Cartan-Ambrose-Hicks**

Let $M, \widetilde{M}$ be Riemannian mfd of dimension n. $p \in M$, $\widetilde{p} \in \widetilde{M}$. $i : T_pM \to T_{\widetilde{p}}\widetilde{M}$ is a linear isometry. $V \subset M$ is a normal neighborhood of p such that $\exp_{\widetilde{p}}$ is defined at $i(\exp_p^{-1}(V))$. Define $f : V \to \widetilde{M}$ by

$$f(q) = \exp_{\widetilde{p}} \circ i \circ \exp_p^{-1}(q).$$

For any $q \in V$ there is unique normalized geodesic $\gamma : [0,t] \to M$ starting at p ending at q. Let P_t be the parallel transport along γ from 0 to t. Define $\phi_t : T_qM \to T_{f(q)}\widetilde{M}$ by

$$\phi_t(v) = \widetilde{P}_t \circ i \circ P_t^{-1}(v).$$

If for all $q \in V$ and all $x, y, u, v \in T_qM$ we have

$$(R(x,y)u, v) = \Big(\widetilde{R}\big(\phi_t(x), \phi_t(y)\big)\phi_t(u), \phi_t(v)\Big),$$

then $f : V \to f(V)$ is an isometry with $df_p = i$.

8.4.1 Theorem

M^n is a complete Riemannian manifold, with constant sectional curvature K. Then the universal cover of M is isometric to
(a) H^n if $K = -1$;
(b) $\mathbb{R}^n$ if $K = 0$;
(c) S^n if $K = 1$.

Complete manifolds with constant sectional curvature are called space forms.

8.4.4 Proposition

Complete Riemannian manifold of even dimension are the sphere and real projective space.

8.4.5 Proposition

Every compact orientable surface of genus $p > 1$ can be provided with a metric of constant negative curvature.

9.2.- (Definition)

Let $f : (-\epsilon, \epsilon) \times [0, a] \to M$ be a variation of c.
Define length function $L : (-\epsilon, \epsilon) \to \mathbb{R}$ by

$$L(s) = \int_0^a \left| \frac{\partial f}{\partial t}(s,t) \right| dt, \quad s \in (-\epsilon, \epsilon).$$

Define energy function $E : (-\epsilon, \epsilon) \to \mathbb{R}$ by

$$E(s) = \int_0^a \left| \frac{\partial f}{\partial t}(s,t) \right|^2 dt, \quad s \in (-\epsilon, \epsilon).$$

Then by Schwarz inequality, $L(c)^2 \le aE(c)$.

9.2.3 Lemma

Minimizing geodesics are the only ones to minimize E.

9.2.4 Proposition

Let $f : (-\epsilon, \epsilon) \times [0, a] \to M$ be a variation of c. Then

$$\begin{aligned} \frac{1}{2}E'(0) &= -\int_0^a \left(V(t), \frac{D}{dt}\frac{dc}{dt} \right) dt - \sum_{i=1}^k \left(V(t_i), \frac{dc}{dt}(t_i^+) - \frac{dc}{dt}(t_i^-) \right) \\ &\quad - \left(V(0), \frac{dc}{dt}(0) \right) + \left(V(a), \frac{dc}{dt}(a) \right). \end{aligned}$$

Here V is the variational field.

9.2.8 Proposition

γ is a geodesic, and $f : (-\epsilon, \epsilon) \times [0, a] \to M$ is a proper variation of γ. Then

$$\begin{aligned} \frac{1}{2}E''(0) &= -\int_0^a \left(V(t), \frac{D^2V}{dt^2} + R(\frac{d\gamma}{dt}, V)\frac{d\gamma}{dt} \right) dt \\ &\quad - \sum_{i=1}^k \left(V(t_i), \frac{DV}{dt}(t_i^+) - \frac{DV}{dt}(t_i^-) \right). \end{aligned}$$

If the variation f is not proper, then the following should be added to the right-hand side:

$$-\left(\frac{D}{dx}\frac{\partial f}{\partial s}, \frac{d\gamma}{dt} \right)_{(0,0)} + \left(\frac{D}{dx}\frac{\partial f}{\partial s}, \frac{d\gamma}{dt} \right)_{(0,a)} - \left(V(0), \frac{DV}{dt}(0) \right) + \left(V(a), \frac{DV}{dt}(a) \right).$$

So

$$\begin{aligned} \frac{1}{2}E''(0) &= \int_0^a [(V', V') - (R(\gamma', V)\gamma', V)]dt \\ &\quad - \left(\frac{D}{dx}\frac{\partial f}{\partial s}, \frac{d\gamma}{dt} \right)_{(0,0)} + \left(\frac{D}{dx}\frac{\partial f}{\partial s}, \frac{d\gamma}{dt} \right)_{(0,a)}. \end{aligned}$$

9.2.- (Notation)

Write

$$\int_0^a \big[(V', V') - (R(\gamma', V)\gamma', V)\big]dt = I_a(V, V).$$

If the variation is proper, $I_a(V, V) = \frac{1}{2}E''(0)$.

9.3.1 Theorem (**Bonnet-Myers**)

M^n is complete. The Ricci curvature of M satisfies

$$\mathrm{Ric}_p(v) \geq \frac{1}{r^2} > 0,$$

for all $p \in M$, all $v \in T_pM$. Then M is compact and $\mathrm{diam}(M) \leq \pi r$.

9.3.2 Corollary

As above. The universal cover of M is compact, and the fundamental group is finite.

9.3.7 Theorem (**Weinstein-Synge**)

f is an isometry of compact oriented M^n, and M has positive sectional curvature. Then f has a fixed point if
(1) n is even, and f preserves orientation; or
(2) n is odd, and f reverses orientation.

9.3.10 Corollary (**Synge**) M^n is compact with positive sectional curvature; then
(1) if M is orientable and n is even, then M is simply connected;
(2) if n is odd, then M is orientable.

10.2.2 **Index Lemma**

$\gamma : [0, a] \to M$ is geodesic without conjugate point. Let J be a Jacobi field, with $(J, \gamma') = 0$. Let V be a piecewise differentiable vector field along γ, with $(V, \gamma') = 0$. Suppose $J(0) = V(0) = 0$ and $J(a) = V(a)$. Then, $I_a(J, J) \leq I_a(V, V)$. Equality occurs iff $V \equiv J$.

10.2.3 Theorem (**Rauch Comparison**)

Let $\gamma : [0, a] \to M^n$ and $\tilde{\gamma} : [0, a] \to \tilde{M}^{n+k}$ be geodesics with same velocity. Let $J, \tilde{J}$ be Jacobi fields along $\gamma, \tilde{\gamma}$, such that

$$J(0) = 0 = \tilde{J}(0), \quad (J'(0), \gamma'(0)) = (\tilde{J}'(0), \tilde{\gamma}'(0)), \quad |J'(0)| = |\tilde{J}'(0)|.$$

Assume $\tilde{\gamma}$ has no conjugate points on $(0, a]$, and for all t and all $x \in T_{\gamma(t)}M$, $\tilde{x} \in T_{\tilde{\gamma}(t)}\tilde{M}$, we have $\tilde{K}(\tilde{x}, \tilde{\gamma}'(t)) \geq K(x, \gamma'(t))$.

Then, $|\tilde{J}| \leq |J|$.

In addition, if $|\tilde{J}| = |J|$ at some $t_0 \in (0, a]$, then

$$\tilde{K}(\tilde{x}, \tilde{\gamma}'(t)) = K(x, \gamma'(t)) \quad \forall t \in [0, t_0].$$

10.2.4 Proposition

Suppose the sectional curvature K of M satisfies

$$0 < L \leq K \leq H$$

where H, L are constants. Then the distance d between any pair of conjugate points along the geodesic satisfies

$$\frac{\pi}{\sqrt{H}} \leq d \leq \frac{\pi}{\sqrt{L}}.$$

10.2.5 Proposition

$p \in M^n$, $\tilde{p} \in \tilde{M}^n$. All sectional curvatures of $\tilde{M}$ greater than or equal to all sectional curvatures of M. Fix a linear isometry$i : T_pM \to T_{\tilde{p}}\tilde{M}$. Let $r > 0$ be such that the restriction $\exp_p|_{B_r(0)}$ is a diffeomorphism and $\exp_{\tilde{p}}|_{\tilde{B}_r(0)}$ is non-singular. Let c be a smooth curve in $\exp_p(B_r(0)) \subset M$, and let $\tilde{c}$ be the curve in $\tilde{M}$

$$\exp_{\tilde{p}} \circ i \circ \exp_p^{-1} \circ c.$$

Then $L(c) \geq L(\tilde{c})$.

10.2.- Remark

M^n, sectional curvature $\leq K_0$, where constant $K_0 > 0$. Then $\exp_p : T_pM \to M$ has no critical point in the open ball of radius $\frac{\pi}{\sqrt{K_0}}$, centered at p.

10.3.1 Theorem (Moore)

Let $\overline{M}$ be a complete simply connected Riemannian manifold, with sectional curvature

$$\overline{K} \leq b \leq 0.$$

M a compact Riemannian manifold whose sectional curvature has $K - \overline{K} \leq -b$. If $\dim\overline{M} < 2\dim M$, then there cannot be an immersion $f : M \to \overline{M}$.

10.4.- **Toponogov Theorem**

M complete, sectional curvature $\sec \geq H$.

1. (Hinge Version) γ_1 and γ_2 normalized geodesics with $\gamma_1(0) = \gamma_2(0)$. Let M_H^2 be the 2-dim space form with curvature H. Assume γ_1 is minimizing, and $l(\gamma_2) \leq \frac{\pi}{\sqrt{H}}$ if $H > 0$. Let $\bar{\gamma}_1, \bar{\gamma}_2$ be normalized geodesics of M_H^2 with $\bar{\gamma}_1(0) = \bar{\gamma}_2(0)$, $l(\gamma_i) = l(\bar{\gamma}_i)$, and the angle in M is the same as in M_H^2. Then, $d(\gamma_1(l_1), \gamma_2(l_2)) \leq d(\bar{\gamma}_1(l_1), \bar{\gamma}_2(l_2))$.

2. (Triangle Version) $(\gamma_1, \gamma_2, \gamma_3)$ determine a geodesic triangle ($\gamma_i(l_i) = \gamma_{i+1}(0)$, and they satisfy triangle inequality). Suppose γ_1, γ_3 are minimizing, and if $H > 0$, $L(\gamma) \leq \frac{\pi}{\sqrt{H}}$. Then in M_H^2, there is a geodesic triangle $(\overline{\gamma}_1, \overline{\gamma}_2, \overline{\gamma}_3)$ with the same lengths as in M, and the angle between $\overline{\gamma}_1, \overline{\gamma}_2$ is smaller than that in M_H^2. Also the triangle in M_H^2 is uniquely determined, unless $H > 0$ and $L(\gamma_i) = \frac{\pi}{\sqrt{H}}$ for some i.

-.-.- **Gauss-Bonnet Theorem**

M is a compact oriented surface, then

$$\int_M K = 2\pi \cdot \chi(M).$$

Here χ is the Euler characteristic. $\chi = 2 - 2g$ if orientable, $2 - g$ if not.

10.-.- **Klingenberg's Lemma**

M complete, $\sec \leq K_0$ where $K_0 > 0$ is constant. γ_0, γ_1 two geodesics joining p, q. γ_0, γ_1 are homotopic with endpoints fixed, via γ_t where $t \in [0, 1]$. Then $\exists t_0 \in [0, 1]$ such that $l(\gamma_0) + l(\gamma_{t_0}) \geq \frac{2\pi}{\sqrt{K_0}}$.

11.2.1 **Index Form**

$\gamma : [0, a] \to M_n$ is geodesic. $\mathcal{V}(0, a)$ is the vector space, of vector fields V along γ which are piecewise C_∞ and vanish at 0 and a.

Then the index form of γ is the symmetric bilinear form I_a defined on $\mathcal{V}$ by

$$I_a(V, W) = \int_0^a \Big[(V', W') - (R(\gamma', V)\gamma', W)\Big] dt.$$

The nullity of B is the dimension of subspace of $\mathcal{V}$ formed by such V that $B(V, -) \equiv 0$.

The index is the maximal dimension of subspaces of $\mathcal{V}$ on which B is negative definite.

11.2.2 **Morse Index Theorem**

The index of I_a is equal to the number of points $\gamma(t)$, $0 < t < a$, conjugate to $\gamma(0)$, each counted with multiplicity. It is finite.

11.2.3 Proposition

V belongs to the null space of I_a iff V is a Jacobi field.

11.2.- (Remark)

The index of I_a is a step function, right continuous. At every jump, the difference is the multiplicity of the conjugate point.

11.- (Exercise)

A line in a complete M, is a geodesic $\gamma : \mathbb{R} \to M$ which minimizes the arc length between any two of its points. Then, if sec > 0, then M contains no lines.

12.3.2 **Preissman Theorem**

M compact with curvature < 0. Then any abelian subgroup of $\pi_1(M)$ is $\mathbb{Z}$ or 0. Also, $\pi_1(M) \neq \mathbb{Z}$.

12.3.10 **Byers Theorem**

M compact with curvature < 0. Then any solvable subgroup of $\pi_1(M)$ is $\mathbb{Z}$ or 0. And $\pi_1(M)$ have no cyclic subgroup of finite index.

13.2.- Cut Point, Cut Locus

$\gamma : [0, \infty) \to M$, normalized geodesic. Then $d(\gamma(0), \gamma(t)) = t$ holds in $[0, t_0)$, but not in (t_0, ∞). $\gamma(t_0)$ is the cut point of $\gamma(0)$ along γ.

The locus of cut points is cut locus, denoted $C_m(p)$.

13.2.2 Proposition

$\gamma(t_0)$ is the cut point of $p = \gamma(0)$. Then:

(a) either $\gamma(t_0)$ is the first conjugate point along γ,

(b) or there is geodesic $\sigma \neq \gamma$ from p to $\gamma(t_0)$, with same length.

Conversely, is (a) or (b) happens for t_0, then the cut point of γ happens at or before t_0.

13.2.7 Corollary

$q \in C_m(p)$, then $p \in C_m(q)$. (similarly for conjugate locus $C(p)$.)

13.2.8 Corollary

If p is not in $C_m(p)$, then there is unique minimizing geodesic joining p, q.

13.2.- **Injectivity Radius**

On T_pM, within some open ball, $\exp_p$ is injective. $i(M)$ is the max radius of the ball.

$$i(M) = \inf_{p \in M} d(p, C_m(p)).$$

13.2.9 Theorem

The distance to $C_m(-)$ is a continuous function from T_M to $\mathbb{R}^+ \cup \{\infty\}$.

Therefore, $C_m(p)$ is closed.

13.2.12 Proposition

$q \in C_m(p)$ realizes the distance from p to $C_m(p)$, then

(a) either q is the first conjugate point along some geodesic,

(b) or there are exactly two minimizing geodesic γ, σ from p to q, in addition, the two geodesics have angle π at q.

13.2.13 Proposition

M complete with sectional curvature K, with $0 < K_{\min} \leq K \leq K_{\max}$. Then:

(a) $i(M) \geq \frac{\pi}{\sqrt{K_{\max}}}$, or

(b) there is closed geodesic γ, whose length is minimal, with $i(M) = \frac{l(\gamma)}{2}$.

13.3.1 Proposition (Klingenberg)

M^n, $n \geq 3$, $\pi_1(M) = 0$, compact, with $\frac{1}{4} < \sec < 1$.

Then, the injectivity radius $i(M) \geq \pi$.

13.3.4 Proposition (Klingenberg)

M^{2n}, orientable, compact, with $0 < \sec < 1$.

Then, the injectivity radius $i(M) \geq \pi$.

13.4.1 Lemma (Berger)

M compact, and $d(p, q) = \text{diam}(M)$. Then $\forall W \in T_pM$, there is a minimizing geodesic γ from p to q with $(\gamma'(0), W) \geq 0$. (q is a critical point of distance function d_p.)

13.4.2 Lemma

M compact, $\pi_1 = 0$, with $\frac{1}{4} < \delta \leq \sec \leq 1$. Let $d(p, q) = \text{diam}(M)$. Then for any $\rho \in (\frac{\pi}{2\sqrt{\delta}}, \pi)$,

$$M = B_\rho(p) \cup B_\rho(q).$$

13.1.1 **Sphere Theorem**

M^n compact, $\pi_1 = 0$, with $\frac{1}{4}K_0 < \sec \leq K_0$.

Then M is homeomorphic to S^n.

Gauss Map

$f : M^n \to \mathbb{R}^{n+1}$ Riemannian immersion.

By possibly passing to the orientation cover of M, we can assume that a unit normal field exists globally.

Let $N : M \to \mathbb{R}^{n+1}$ be such a choice of unit normal field. Then we obtain the gauss map

$$G : M \to S^n \subset \mathbb{R}^{n+1}.$$

Then, $DG(v) = -S_N(v)$ where S is the shape operator, because $S_N(v) = -\nabla_v N$.

Theorem (Hadamard)

Let $f : M^n \to \mathbb{R}^{n+1}$ be isometric immersion, $n > 1$. If the shape operator is always positive, then M is diffeomorphic to S^n via the Gauss map, and f is an embedding.

Hessian

$f : M^n \to \mathbb{R}$ is smooth. Define the Hessian, Hess(f) at p, as the self-adjoint linear operator

$$\mathrm{Hess} f : T_pM \to T_pM, \quad (\mathrm{Hess} f)Y \triangleq \nabla_Y \mathrm{grad} f.$$

Or, we can regard it as a symmetric 2-tensor

$$\mathrm{Hess} f(X, Y) = (\nabla_X \mathrm{grad} f, Y) = X(Yf) - (\nabla_X Y)f.$$

Then the Laplacian $\triangle f$ is given by

$$\triangle f = \mathrm{div}(\mathrm{grad} f) = \mathrm{tr}(\mathrm{Hess} f).$$

The Hessian coincides with second fundamental form of the level sets of f, with normal vector the gradient.

Riemannian Submersion

$f : M^{n+k} \to N^n$ is a submersion (surjective on tangent spaces). For any $p \in M$, $v \in T_pM$, call v vertical if $df_p(v) = 0$ in N, and call v horizontal if it is orthogonal to the fiber (*i.e.* orthogonal to all vertical vectors in T_pM). Denote the vertial space V_p, and the horizontal space H_p. They are distributions on M.

The submersion is called a Riemannian submersion, if $df_p : T_pM \to T_{f(p)N}$ restricted to H_p, is an isometry for any $p \in M$.

Soppose f is a Riemannian submersion. Given $X \in \mathfrak{X}(N)$, there is a unique horizontal lift $\overline{X} \in \mathfrak{X}(M)$, such that $\overline{X}$ is horizontal, and it is f-related to X.

Lie Derivative

$X, Y_1, \ldots, Y_k \in \mathfrak{X}(M)$, and σ a k-tensor field (in particular, σ can be a differential form) on M. Then the Lie derivative of σ is

$$\begin{aligned}(L_X\sigma)(Y_1, \ldots, Y_k) &= X(\sigma(Y_1, \ldots, Y_k)) \\ &\quad -\sigma([X, Y_1], Y_2, \ldots, Y_k) - \cdots - \sigma([Y_1, \ldots, Y_{k-1}, [X, Y_k]).\end{aligned}$$

Cartan's Formula

For any vector field X and differential form ω on M (just a smooth manifold, not necessarily with Riemannian metric),

$$L_X\omega = i(X)(d\omega) + d(i(x)\omega) = d\omega(X, -, \cdots, -) + d(\omega(X, -, \cdots, -)).$$

Proposion

$f : M \to N$ Riemannian submersion. $X, Y, Z \in \mathfrak{X}(N)$, with $\overline{X}, \overline{Y}, \overline{Z}$ on M. V is a vertical vector field on M. Then:

(1) $[V, \overline{X}]$ is vertical;

(2) $(L_V\bar{g})(\overline{X}, \overline{Y}) = (D_V\bar{g})(\overline{X}, \overline{Y}) = 0$, here L is the Lie derivative;

(3) $\bar{g}([\overline{X}, \overline{Y}], V) = 2\bar{g}(\nabla_{\overline{X}}\overline{Y}, V) = -2\bar{g}(\nabla_V\overline{X}, \overline{Y}) = 2\bar{g}(\nabla_{\overline{Y}}V, \overline{X})$;

(4) $\nabla_{\overline{X}}\overline{Y} = \overline{\nabla_X Y} + \frac{1}{2}[\overline{X}, \overline{Y}]^v$, here $(-)^v$ means decomposing the vector into vertial and horizontal parts, and taking only the vertical component.

Integrability Tensor

The map $H_p \times H_p \to V_p$ defined by $(X, Y) \mapsto [X, Y]^v$ is a skew-symmetric 2-tensor, called the integrability tensor. It measurs the extent to which the horizontal distribution is integrable, in the sense of Frobenius.

O'Neil Formula

If $f : M \to N$ Riemannian submersion, and X, Y, H, Z vector fields on N

with zero Lie bracket, then

$$\begin{aligned}\bar{R}(\bar{X},\bar{Y},\bar{H},\bar{Z}) &= R(X,Y,H,Z)-\frac{1}{4}([\bar{Y},\bar{Z}],[\bar{X},\bar{H}]) \\ &\quad +\frac{1}{4}([\bar{X},\bar{Z}],[\bar{Y},\bar{H}])-\frac{1}{2}([\bar{X},\bar{Y}],[\bar{H},\bar{Z}]).\end{aligned}$$

Therefore,

$$R(X,Y,X,Y)=\bar{R}(\bar{X},\bar{Y},\bar{X},\bar{Y})+\frac{3}{4}\left|[\bar{X},\bar{Y}]^v\right|^2.$$

Relative Volume Comparison, Bishop-Cheeger-Gromov

M complete, with $\mathrm{Ric}\geq k$, then the function

$$r\mapsto\frac{\mathrm{vol}\big(B_q(r)\big)}{v(n,k,r)}$$

is non-increasing, and its $\lim_{r\to 0^+}$ is 1.

Here $v(n,r,k)$ is the volume of a ball of radius r in the constant-curvature k space form S^n_k.

Maximal Diameter Rigidity (S. Y. Cheng)

M complete, $\mathrm{Ric}\geq k$, $\mathrm{diam}=\frac{\pi}{k}$. Then M is S^n_k.

Soul Theorem (Cheeger-Gromoll-Meyer)

M is complete, non-compact, $\sec\geq 0$. Then M contains a soul S, which is a compact, totally convex (and totally geodesic) submanifold, such that M is diffeomorphic to the normal bundle over S. Any two souls are isomorphic.

If $\sec>0$, the soul is a point, and M is diffeomorphic to $\mathbb{R}^n$.

Finiteness of π_1

There is constant c_n, such that any complete M^n with $\sec\geq 0$ satisfies: $\pi_1(M)$ can be generated by $\leq c_n$ generators.

Finiteness of Betti Numbers

There is constant c_n, such that any complete M^n with $\sec\geq 0$ satisfies: for any field F,

$$\sum_{i=0}^n b_i(M;F)=\sum_{i=0}^n \mathrm{dim}H_i(M;F)\leq c_n.$$

Splitting Theorem (Cheeger-Gromoll)

M contains a line, and M has Ricci curvature ≥ 0, then M is isometric to some $H \times \mathbb{R}$ with product metric.

Critical Point of Distance Function

$A \subset M$ is closed, define distance function $r : M \to \mathbb{R}$ to be $r(p) = d(p, A)$. Point p is called a critical point of r if: for any $v \in T_pM$, there is a minimizing geodesic c from p to A with $(\dot{c}(0), v) \geq 0$. A point not critical, is called regular. Point p is regular iff the initial tangent vectors of minimizing geodesics from p to A all lie in one open half-space.

For any angle α, point p is called α-regular if: there is $v \in T_pM$ such that the angle between v and the initial tangent vector of any minimizing geodesics from p to A has angle $< \alpha$. Then regular $= \pi/2$-regular.

Proposion

(1) The set of $\alpha-$regular points is open.

(2) In the definition of $\alpha-$regularity, the set of possible vectors $v \in T_pM$ is convex, for all $\alpha \in (0, \pi/2]$.

(3) U is any open set of α-regular points of r, then there is a unit vector field X on U such that X_p can be v above. And, for any integral curve γ of X, and $s < t$,

$$r(\gamma(t)) - r(\gamma(s)) > \cos(\alpha) \cdot (t - s).$$

Berger's Lemma

Any local maximum of distance function $d(-, A)$, is critical. (But not vice versa.)

1st Variation of Arc Length

$\Phi : [0,l] \times (-\epsilon, \epsilon) \to M$ is a variation of γ (parameterized by arc-length, not necessarily geodesic), then

$$L'(0) = (\gamma', U)|_0^l - \int_0^l (U, \nabla_{\gamma'}\gamma')\mathrm{d}t.$$

Here U is the variation field. So when γ is a geodesic, then

$$L'(0) = (\gamma', U)|_0^l.$$

2nd Variation of Arc Length

$\Phi : [0,l] \times (-\epsilon, \epsilon) \to M$ is a variation of geodesic γ (parameterized by arc-length), then

$$L''(0) = (\nabla_U U, \gamma')|_0^l + \int_0^l \left[|U^\perp|^2 + (R(\gamma', U^\perp)\gamma', U^\perp) \right] \mathrm{d}t.$$

List of Knowledge of complete M: $\sec > 1$

Bonnet-Myers:

(1) Compact universal cover, finite π_1.

(2) Diameter$\leq \pi$, with equality holding iff $S^n(1)$.

Weinstein-Synge:

(3) If n odd, then orientable.

(4) If n even and orientable, then $\pi_1 = 0$.

Klingenberg:

(5) If n even and orientable, then $i(M) \geq \frac{\pi}{\sqrt{\max \sec}}$.

(6) If n even and non-orientable, then $i(M) \geq \frac{\pi}{2 \cdot \sqrt{\max \sec}}$.

(7) If n odd, $\pi_1 = 0$, and $\max \sec < 4$, then $i(M) \geq \frac{\pi}{\sqrt{\max \sec}}$.

Sphere Theorem (Berger, Klingenberg, Chen)

(8) If $\pi_1 = 0$ and $\max \sec < 4$, then M is diffeomorphic to S^n.