Math 202 | Exam 1 | Jerry L. Kazdan
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Exam 1

DIRECTIONS: Part A has 8 shorter problems (5 points each) while Part B has 3 traditional problems (10 points each). To receive full credit your solution should be clear and correct. Neatness counts. You have 1 hour 20 minutes. Closed book, no calculators, but you may use one 3 ×5 with notes on both sides.

PART A: Eight shorter Problems, 5 points each.

A-1. Show that $\sqrt{5}$ is not a rational number.

SOLUTION: Say $\sqrt{5}$ is a rational number, $\sqrt{5} = a/b$, where a and b. We may assume that a and b have no common factors. Now $5b^2 = a^2$ so 5 is a factor of a^2. Since 5 is a prime number, it is a factor of a. Thus $a = 5k$ for some integer k. But then $b^2 = 5k^2$ so we see that b^1 and hence b is divisible by 5. This contradicts that a and b have no common factor.

A-2. If a and b are *rational* numbers, consider the set S of real numbers of the form $a + b\sqrt{5}$. Show that the non-zero elements in S have multiplicative inverses in S. [This is the key step in showing that S is a field.]

SOLUTION: The multiplicative inverse of $a + b\sqrt{5}$ as a real number is $1/(a + b\sqrt{5}$. If a and b are rational we want to write this in the form $\alpha + \beta\sqrt{5}$, where α and β are rational. We use a standard procedure:

$$\frac{1}{a+b\sqrt{5}} = \frac{1}{a+b\sqrt{5}}\left(\frac{a-b\sqrt{5}}{a-b\sqrt{5}}\right) = \frac{a-b\sqrt{5}}{a^2-5b^2} = \left(\frac{a}{a^2-5b^2}\right) + \left(\frac{-b}{a^2-5b^2}\right)\sqrt{5}.$$

The denominator is never zero because $\sqrt{5}$ is irrational.

A-3. Determine if the set $S = \{x \in \mathbb{R} :\ 2x^2 > x^3 - 3x\}$ is bounded above and/or below, and if so, find $\inf(S)$ and $\sup(S)$ — if they exist.

SOLUTION: Rewrite this as $p(x) := x^3 - 2x^2 - 3x < 0$. Factoring the polynomial we find $p(x) = x(x-3)(x+1) < 0$. Clearly $p(x)$ is large positive for x large positive and negative for x large negative. Since we know the roots of p are -1, 0, and 3, we see that p is negative for $x < -1$ and $0 < x < 3$. This is the set S. Its sup is $x = 3$. Because S is unbounded below it has no $\inf$.

A-4. Give an example of a sequence of real numbers that is not monotone but that does converge to some limit.

SOLUTION: $\dfrac{(-1)^n}{n}$

A-5. If x_1 is a given real number and $x_{n+1} = \sqrt{1 + x_n^2}$ for $n = 1, 2, \ldots$, show that the sequence x_n diverges.

SOLUTION: *Method 1.* Compute the first few terms to try to see what is happening.

$x_2 = \sqrt{1+x_1^2}$, $\quad x_3 = \sqrt{1+(1+x_1^2)} = \sqrt{2+x_1^2}$, $\quad x_4 = \sqrt{1+(2+x_1^2)} = \sqrt{3+x_1^2}, \ldots$.
The pattern is clear: $x_{n+1} = \sqrt{n+x_1^2}$ which diverges.

Method 2. Let $u_n := x_n^2$. Then $u_{n+1} = 1+u_n$ so $u_{n+1} = n+u_1$ which is unbounded.

Method 3. Reasoning by contradiction, say $x_n \to L$. Then $L = \sqrt{1+L^2} > L$.

A-6. Let $f,\ g : \mathbb{R} \to \mathbb{R}$ be bounded functions such that $f(x) \le g(x)$ for all x. Let F denote the image of f and G the image of g. Give an example (a picture) of pairs of such functions with $\sup(F) > \inf(G)$.

Solution: A simple example is $f(x) := \cos x$ and $g(x) := \cos x + 1$. Simpler, let $f(x) = g(x) = \cos x$. More generally, $f(x)$ could be any bounded function that is not the constant function and let $g(x) := f(x)$.

A-7. Compute $\lim\limits_{n\to\infty} \dfrac{1+2n-5n^2}{4+3n^2}$. Carefully note any standard theorems you use.

Solution: For large n this fraction is essentially $\frac{-5n^2}{3n^2} = \frac{-5}{3}$. Since this "computation" cancelled infinities from numerator and denominator, a real proof should be more careful. Dividing numerator and denominator by n^2, we want to compute

$$\lim_{n\to\infty}\left(\frac{\frac{1}{n^2}+\frac{2}{n}-5}{\frac{4}{n^2}+3}\right) = \frac{\lim_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n}-5\right)}{\lim_{n\to\infty}\left(\frac{4}{n^2}+3\right)} = \frac{-5}{3}.$$

In this computation we used the theorem that if $a_n \to A$ and $b_n \to B$, then $a_n + b_n \to A+B$, and also $a_n/b_n \to A/B$ (assuming $b_n \ne 0$ and $B \ne 0$).

A-8. Give an example of a sequence x_n of real numbers with at least two subsequences that converge to different limits.

Solution: Example 1). $x_n = (-1)^n$, $\quad$ Example 2). $x_n = (-1)^n + \frac{1}{n}$.

Part B: Three traditional problems, 10 points each.

B-1. a) For which real numbers $c > 0$ does $\lim\limits_{n\to\infty} n^2 c^n = 0$? Why?

Solution: If $c \ge 1$ this clearly diverges to infinity. If, say, $c = 1/2$, then the sequence is $n^2/2^n$ so at each step the denominator is doubled while the numerator increases more slowly. It looks like in this case, the sequence converges to zero.
After this experimentation, the *ratio test* efficiently resolves the issue. Let $a_n = n^2c^n$. Then as n tends to infinity,

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2c^{n+1}}{n^2c^n} = \frac{(n+1)^2}{n^2}c \to c.$$

By the ratio test, if $0 < c < 1$, this sequence converges to 0. If $c \ge 1$, a direct inspection (done above) alredy showed that the sequence diverges.

b) Repeat this if c is a complex number.

SOLUTION: There is essentially no change since we can take absolute values. Here are the details. As above, let $a_n = n^2c^n$. Then $|a_n| = n^2|c|^n$. By part a), if $|c| < 1$, then $|a_n| \to 0$ (and hence $a_n \to 0$). If $|c| \ge 1$, then the sequence clearly blows up.

B-2. Let the real sequence $b_n > 0$ converge to a limit $B > 0$. Show with your bare hands (an ϵ argument) that $1/b_n \to 1/B$.

SOLUTION: Given $\epsilon > 0$ we want an integer $N(\epsilon)$ so that if $n \ge N$, then

$$\left|\frac{1}{n} - \frac{1}{B}\right| < \epsilon, \qquad \text{that is,} \qquad \left|\frac{B - b_n}{b_n B}\right| < \epsilon. \tag{1}$$

There are two issues: keeping the b_n in the denominator away from 0 and making the numerator small. Treat these separately.

Lemma. *If $b_n > 0$ and $b_n \to B > 0$, then there is an N_1 so that if $n \ge N_1$, then $b_n > B/2$.*

PROOF. Since $b_n \to B > 0$, there is an integer N_1 so that if $n \ge N_1$, then $|b_n - B| < B/2$. Thus $-B/2 < b_n - B < B/2$. In particular, $B/2 < b_n$. so $1/b_n < 2/B$.

Using this and keeping (1) in mind, since $b_n \to B$, there is an N so that if $n \ge N$ then $|b_n - B| < \frac{1}{2}B^2\epsilon$. Restricting N further so that $N \ge N_1$, by the Lemma we see that inequality (1) is satisfied:

$$\left|\frac{B - b_n}{b_n B}\right| < \left|\frac{B - b_n}{B^2/2}\right| < \epsilon.$$

B-3. A sequence $x_n \in \mathbb{R}$ is called *contracting* if for some constant $0 < c < 1$ (such as $c = \frac{1}{2}$) it has the property that for all $n = 1, 2, 3, \ldots$

$$|x_{n+1} - x_n| \le c|x_n - x_{n-1}|.$$

The point of this problem is to show that a contracting sequence converges.

a) Show that $|x_{n+1} - x_n| \le c^n|x_1 - x_0|$ for all n.

SOLUTION: Since $|x_2 - x_1| \le c|x_1 - x_0|$, then $|x_3 - x_2| \le c|x_2 - x_1| \le c^2|x_1 - x_0|$. Repeating this we see that $|x_4 - x_3| \le c|x_3 - x_2| \le c^3|x_1 - x_0|$, and, more generally,

$$|x_{n+1} - x_n| \le c^n|x_1 - x_0| \quad \text{for all} \quad n = 0, 1, 2, \ldots.$$

This induction argument is sufficiently obvious that a formal induction proof is not needed.

b) Use $x_{n+1} - x_0 = (x_{n+1} - x_n) + (x_n - x_{n-1}) + \cdots + (x_1 - x_0)$ to show that

$$|x_{n+1} - x_0| \le \left(c^n + c^{n-1} + \cdots + c + 1\right)|x_1 - x_0|$$

SOLUTION: By the triangle inequality and part a),

$$\begin{aligned} |x_{n+1} - x_0| &\le |x_{n+1} - x_n| + |x_n - x_{n-1}| + \cdots + |x_1 - x_0| \\ &\le \left(c^n + c^{n-1} + \cdots + c + 1\right)|x_1 - x_0|. \end{aligned}$$

c) More generally, if $n > k$ show that

$$\begin{aligned} |x_{n+1} - x_k| &\leq \left(c^n + c^{n-1} + \cdots + c^k\right) |x_1 - x_0| \\ &= c^k \left(\frac{1 - c^{n-k+1}}{1 - c}\right) |x_1 - x_0| < c^k \frac{|x_1 - x_0|}{1 - c}. \end{aligned}$$

REMARK: Since $0 < c < 1$, this shows that the x_n are a Cauchy sequence and hence converge.

SOLUTION: This is a straightforward modification of the previous part. By the triangle inequality, part a), and standard formulas for geometric series:

$$\begin{aligned} |x_{n+1} - x_k| &\leq |x_{n+1} - x_n| + |x_n - x_{n-1}| + \cdots + |x_{k+1} - x_k| \\ &\leq \left(c^n + c^{n-1} + \cdots + c^k\right) |x_1 - x_0| \\ &= c^k \left(c^{n-k} + \cdots + c + 1\right))|x_1 - x_0| \\ &= c^k \left(\frac{1 - c^{n-k+1}}{1 - c}\right) |x_1 - x_0| < c^k \frac{|x_1 - x_0|}{1 - c}. \end{aligned}$$

The key point is that the final inequality is independent of n – as long as $n > k$. Since $0 < c < 1$, by choosing k large, then c^k can be as small as you wish.