Math 202 **Exam 2** Jerry L. Kazdan
November 5, 2013 12:00 — 1:20

DIRECTIONS: Part A has 5 shorter problems (8 points each) while Part B has 4 traditional problems (15 points each). [100 points total].
To receive full credit your solution should be clear and correct. Neatness counts. You have 1 hour 20 minutes. Closed book, no calculators, but you may use one $3\ \times 5$ with notes on both sides.

PART A: Five shorter problems, 8 points each [total: 40 points]

A-1. Give an example of an infinite series $\sum a_n$ that converges but does not converge absolutely. [You do not need to justify your assertion.]

SOLUTION: The alternating harmonic series: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots$

A-2. Give an example of a bounded function defined on $-2 \le x \le 2$ that is continuous everywhere *except* at $x = 0$. [You do not need to justify your assertion].

SOLUTION: $f(x) = \begin{cases} 0 & -2 \le x \le 0, \\ 1 & \ \ 0 < x \le 2. \end{cases}$ also $f(x) = \begin{cases} \sin(1/x) & x \ne 0, \\ 0 & x = 0. \end{cases}$.

The function $g(x) = 1/x$ for $x \ne 0$ and $g(0) = 1$ is *not* an example. Although it is certainly not continuous at $x = 0$, it is not bounded – and the problem asks for a bounded function.

A-3. Show that the polynomial $p(x) := x^6 + x^5 - 5$ has at least two *real* zeroes.

SOLUTION: For x near $\pm\infty$, clearly $p(x) > 0$. Also $p(0) = -5$. Now apply the intermediate value theorem to the two intervals $x \le 0$ and $x \ge 0$. [One could also have observed that just as obviously $p(\pm 2) > 0$ so there are two real roots in the interval $-2 < x < 2$].

A-4. Let $g(x)$ be any smooth function and let $f(x) = (x-1)(x-2)(x-3)g(x)$. Show there is a point $c \in (1,\,3)$ where $f''(c) = 0$.

SOLUTION: By Rolle's theorem there are points c_1 with $1 < c_1 < 2$ and c_2 with $2 < c_2 < 3$ with $f'(c_1) = 0$ and $f'(c_2) = 0$. Apply Rolle's theorem again using $f'(x)$ in the interval $[c_1,\,c_2]$ to obtain a point $c \in [c_1,\,c_2]$ with $f''(c) = 0$.

A-5. Say a function $f(x)$ has the properties $f'(x) = 2$ for all $x \in \mathbb{R}$ and $f(1) = 2$. Show that $f(x) = 2x$. [HINT: To show that "$A = B$", it is often easiest to show that "$A - B = 0$".]

SOLUTION: Let $g(x) = f(x) - 2x$. Then $g'(x) = 0$ everywhere. Thus by the Mean Value Theorem $g(x) \equiv$ constant. But $g(1) = 0$ so $g(x) = 0$ everywhere.

Slight variant. Since $f'(x) \equiv 2$, by the Mean Value Theorem, there is a c between 1 and x so that

$$f(x) - f(1) = f'(c)(x-1) = 2(x-1), \qquad \text{that is} \qquad f(x) - 2 = 2(x-1).$$

Thus, $f(x) = 2x$.

PART B: Four traditional problems, 15 points each [60 points[

B-1. Determine if the series $1 + \dfrac{1}{3} + \dfrac{1}{5} + \dfrac{1}{7} + \cdots$ converges or diverges. Please explain your reasoning.

SOLUTION: $1 + \dfrac{1}{3} + \dfrac{1}{5} + \dfrac{1}{7} + \dfrac{1}{9} + \cdots > \dfrac{1}{4} + \dfrac{1}{6} + \dfrac{1}{8} + \dfrac{1}{10} + \cdots = \dfrac{1}{2}\left[\dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \dfrac{1}{5} + \cdots\right]$, so the series diverges by comparison with the harmonic series.

B-2. Use the definition of the derivative as the limit of a difference quotient to show that if $f(x) = \cos 2x$, then f is differentiable everywhere and compute its derivative. [You may use that $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ and $\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} = 0$.]

SOLUTION:

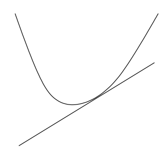
$$\begin{aligned}\frac{\cos 2(x+h) - \cos 2x}{h} &= \frac{[\cos 2x \cos 2h - \sin 2x \sin 2h] - \cos 2x}{h} \\ &= \frac{\cos 2x(\cos 2h - 1)}{h} - \frac{\sin 2x \sin 2h}{h} \\ &= 2\frac{\cos 2x(\cos 2h - 1)}{2h} - 2\frac{\sin 2x \sin 2h}{2h}.\end{aligned}$$

Now let $h \to 0$ and use $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ and $\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} = 0$ to see that

$$\lim_{h\to 0} \frac{\cos 2(x+h) - \cos 2x}{h} = 0 - 2\sin 2x = -2\sin 2x.$$

Thus $\cos 2x$ is differentiable everywhere and its derivative is $-2\sin 2x$.

B-3. Let $f(x)$ have two continuous derivatives in the interval (a, b) and say $f''(x) \geq 0$ for all $x \in [a, b]$. Prove that for any x_0 the graph of $y = f(x)$ lies above its tangent line at $(x_0, f(x_0))$. [If the equation of the tangent line at x_0 is $y = g(x)$, then by "lies above" I mean $f(x) \geq g(x)$ for all $x \in [a, b]$.]



SOLUTION: The equation of the tangent line at x_0 is $g(x) = f(x_0) + f'(x_0)(x - x_0)$.

METHOD 1. Taylor's Theorem says that there is a c between x_0 and x so that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \tfrac{1}{2} f''(c)(x - x_0)^2.$$

Since $f''(c) \geq 0$, this shows that $f(x) \geq g(x)$ for all x, that is, the curve lies above its tangent line.

METHOD 2. If $x > x_0$, then there is some $p \in (x_0, x)$ where $f(x) - f(x_0) = f'(p)(x - x_0)$. But because $f''(x) \geq 0$, we know that $f'(p) \geq f'(x_0)$. Thus $f(x) - f(x_0) \geq f'(x_0)(x - x_0)$, that is, $f(x) \geq f(x_0) + f'(x_0)(x - x_0)$, as desired.

Similarly, If $x \leq x_0$, then there is some $q \in (x, x_0)$ where $f(x_0) - f(x) = f'(q)(x_0 - x)$. Since $f'(q) \leq f'(x_0)$, then $f(x_0) - f(x) \leq f'(x_0)(x_0 - x)$, that is, $f(x) \geq f(x_0) + f'(x_0)(x - x_0)$, as desired.

B-4. Suppose a function $f : \mathbb{R} \to \mathbb{R}$ has the property that there is a constant $a > 0$ so that $f'(x) \geq a$ for all $x \in \mathbb{R}$.

a) Show that if $x \geq 0$, then $f(x) \geq f(0) + ax$ while if $x \leq 0$, then $f(x) \leq f(0) + ax$.

SOLUTION: If $x \geq 0$, by the Mean Value Theorem there is a c_1 in the interval $(0, x)$ where

$$f(x) - f(0) = f'(c_1)x \geq ax, \quad \text{, that is,} \quad f(x) \geq f(0) + ax.$$

Similarly, if $x \leq 0$, by the Mean Value Theorem there is a $c_2 \in (x, 0)$ where

$$f(x) - f(0) = f'(c_2)x \leq ax, \quad \text{, that is,} \quad f(x) \leq f(0) + ax.$$

b) Show that for every $c \in \mathbb{R}$ there is one (and only one) solution of the equation

$$f(x) = c.$$

Thus, there are two steps: (i) show the equation has at least one solution and (ii) show that the equation has at most one solution.

[NOTE The existence of at least one solution may be *false* if you assume only $f'(x) > 0$. For example the equation $e^x = -1$ has no solution.]

SOLUTION: By part a) we see that as $x \to +\infty$ then $f(x) \to +\infty$ and as $x \to -\infty$ then $f(x) \to -\infty$ (this was the point for including Part (a)). Thus by the Intermediate Value Theorem for any c there is at least one x such that $f(x) = c$.

Next we show there is at most one such solution. Reasoning by contradiction, say there were two distinct solutions, $x_1 < x_2$. Then $f(x_1) = f(x_2) = c$. But by the Mean Value Theorem there is a point γ between c_1 and c_2 where $f(x_2) - f(x_1) = f'(\gamma)(x_2 - x_1) > 0$, a contradiction. [This uniqueness part only used $f'(x) > 0$, not $f'(x) \geq a > 0$.]