Math 202, Fall 2013 Jerry L. Kazdan

Notes on the Solution of $x^2 = 2$

These are my notes that the equation $x^2 = 2$ has a real solution. They rely critically on the *Archimedian property of the real numbers*: If a and b are any two positive real numbers, then there is a positive integer n such that $na > b$. Equivalently, there is an integer such that $b/n < a$.

Let $S = \{x \in \mathbb{R} : 0 < x^2 < 2\}$. Since $1 \in S$, the set S is not empty. The squares of elements in S are all less than 2 so S is bounded above. Thus, by the least upper bound property of the real numbers, there is a real α that is the least upper bound of S.

I claim that $\alpha^2 = 2$, so α is the desired solution. I demonstrate this by showing that both other possibilities, $\alpha^2 < 2$ and $\alpha^2 > 2$, give contradictions.

Case 1 Intuitive idea: If $\alpha^2 < 2$, it looks like it is too small so we will increase it a bit to obtain a β that is larger than α yet still in S. This will show that in this case α is not an upper bound for S.

Details. Naively seek β in the form $\beta := \alpha + \frac{1}{n}$ and pick n to be a sufficiently large integer. Clearly $\beta > \alpha$. We want to pick n so that $\beta^2 < 2$, since then $\beta \in S$. Because $1/n^2 \le 1/n$, we have

$$\beta^2 = \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} \le \alpha^2 + \frac{2\alpha + 1}{n}.$$

Because $2 - \alpha^2 > 0$, we can now pick n so large that $\frac{2\alpha+1}{n} < 2 - \alpha^2$. This gives $\beta^2 < \alpha^2 + (2 - \alpha^2) = 2$. Consequently β is an element of S that is larger than α and hence α is not an upper bound for S.

Case 2 Intuitive idea: If $\alpha^2 > 2$, it looks like it is too large so we will decrease it a bit to obtain a β that is smaller than α yet still $\beta^2 > 2$ so it is still an upper bound for S. This will show that in this case α is *not* the least upper bound for S.

Details. Naively seek β in the form $\beta := \alpha - \frac{1}{n}$ and pick n to be a sufficiently large integer. Clearly $\beta < \alpha$. We want to pick n so that $\beta^2 > 2$, since then $\beta \notin S$. Now

$$\beta^2 = \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} > \alpha^2 - \frac{2\alpha}{n} = 2 + [\alpha^2 - 2] - \frac{2\alpha}{n}.$$

Because $\alpha^2 > 2$, if we pick n sufficiently large then the right hand side above will be larger than 2, that is, $\beta^2 > 2$ so β is an upper bound for S. But $\beta < \alpha$ so in this case α is not the least upper bound for S.

The following problems use similar ideas and will be on Homework Set 3.

1. Given any rationals p and q with $p < q$, show there is an irrational number α so that $p < \alpha < q$; "between any two rationals there is an irrational."

2. Given any irrationals α and β with $\alpha < \beta$, show there is a rational number r so that $\alpha < r < \beta$.; "between any two irrationals there is a rational."

[Last revised: August 30, 2018]