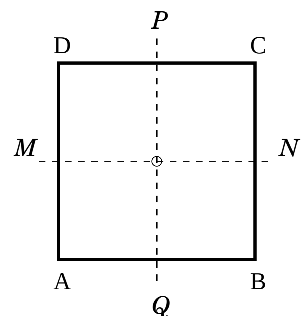
Symmetries of a Square

Example To describe the symmetries of a square $ABCD$, introduce coordinates so that the center of the square is at the origin. One obvious symmetry is a 90 degree counterclockwise rotation R. Then R^2 (just repeat R) is the rotations by 180 degrees. Also R^3 is the rotation by 270 degrees – which is clearly equivalent to a *clockwise* rotation by 90 degrees, which we write as $R^{-1} = R^3$. A rotation by 360 degrees is the same as no rotation, so R^4 is the identity matrix: $R^4 = I$. Observe $R^{-1}R = R^3R = R^4 = I$, as one should want.



Another evident symmetry is the reflection, S, across the vertical line PQ. Clearly reflecting twice brings you back home, so $S^2 = I$.

We can use a sequence of these symmetries, such as SR (a rotation R followed by a reflection S), to get the complete *group of symmetries of the square*. The complete list of elements of this group are:

$$I, \quad R, \quad R^2, \quad R^3, \quad S, \quad SR, \quad SR^2, \quad SR^3. \tag{1}$$

Note that by a computation, $S^2 = I$, $RS = SR^3$, $R^2S = SR^2$, and $R^3S = SR$ so the above list contains all possible combinations of products of R's and S's. Since $SR \neq RS$, this group of symmetries is *not* commutative.

There are some additional evident symmetries of the square, for example the reflection T across the horizontal line MN. Is this missing from our list (1)? If you sketch the figures, you will see that you can achieve T by first using the reflection S followed by R^2. Thus, $T = R^2S$. Similarly, the reflection across the diagonal DB is equivalent to RS. The list (1) really does contain *all* the symmetries of the square.

Exercise:

a) Use $RS = SR^3$ to show that the maps RSR, R^2S, and RSR^{-1} are in the list (1).

b) Prove that the list (1) really does contain *all* the symmetries of the square. I suggest beginning with the special case where the vertex A

is fixed. What are the possible adjacent vertices? A key ingredient is that the symmetries of the square are *rigid motions*, that is, they preserve distances between points, so no stretching or shrinking is allowed.