Waitin' for the bus outta here

In a certain town there are N bus companies whose buses stop at the Main Street station. For each $i = 1, \ldots, N$, the ith company's bus arrives at this station every k_i minutes, but the times of arrival of the previous busses are unknown. The question is, what is the average length of time that you will wait for a bus after arriving at the station?

1 The solution

We number the k_i's in increasing order, $k_1 \leq k_2 \leq \cdots \leq k_N$. Let the random variable X_i denote the time between your arrival at the station and arrival of the next bus of the ith company. The cumulative distribution function (cdf) of X_i is

$$F_i(t) \stackrel{\text{def}}{=} \text{Prob}(X_i \leq t) = \begin{cases} 0, & \text{if } t < 0; \\ t/k_i, & \text{if } 0 \leq t \leq k_i; \\ 1, & \text{if } t > k_i. \end{cases}$$

The probability that *all* of the X_i's are $\geq t$ is

$$\prod_{j=1}^{N} (1 - F_j(t)),$$

which is 0 for $t > k_1$, reflecting the fact that you will surely not wait more than k_1 minutes. Hence the probability that at least one of the X_i's is $\leq t$ is

$$1 - \prod_{j=1}^{N} (1 - F_j(t)).$$

The waiting time that we actually experience at the bus stop will be $X = \min_i X_i$, and the cdf of this random variable is

$$F(t) \stackrel{\text{def}}{=} \text{Prob}(X \leq t) = \begin{cases} 0, & \text{if } t < 0; \\ 1 - \prod_{j=1}^{N} \left(1 - \frac{t}{k_j}\right), & \text{if } 0 < t < k_1; \\ 1, & \text{if } t > k_1. \end{cases}$$

Finally the expected waiting time is

$$E(\mathbf{k}) = \int_{-\infty}^{\infty} t \, dF(t) = k_1 - \int_0^{k_1} F(t) dt = k_1 - \int_0^{k_1} \left(1 - \prod_{j=1}^{N} \left(1 - \frac{t}{k_j}\right)\right) dt. \qquad (1)$$

Here are some examples.

$$
\begin{aligned}
E(a) &= \frac{1}{2}a \\
E(a,b) &= \frac{1}{2}a - \frac{1}{6}\frac{a^2}{b} \\
E(a,a) &= \frac{1}{3}a
\end{aligned}
$$

$$
\begin{aligned}
E(a,b,c,d,e) = {} & -1/42\,\frac{a^6}{bcdef} + 1/30\,\frac{(b+e+d+c+f)\,a^5}{bcdef} \\
& -1/20\,\frac{(eb+db+cd+ef+bc+de+fd+fb+fc+ec)\,a^4}{bcdef} \\
& +1/12\,\frac{(deb+efc+fdb+efb+cde+def+ebc+bcd+fbc+fcd)\,a^3}{bcdef} \\
& -1/6\,\frac{(bcde+fbcd+efbc+defb+cdef)\,a^2}{bcdef} + 1/2\,a
\end{aligned}
$$