Math 313/515, Spring 2005 Jerry L. Kazdan

Homework Set 5, Due Thursday, Feb. 17, 2005

(*Late papers will be accepted until 4 PM on Fri. Feb. 18*)

1. a) With the usual Euclidean norm in $\mathbb{R}^n$ show that for any vectors X, Y:

$$\|X+Y\|^2+\|X-Y\|^2=2\|X\|^2+2\|Y\|^2.$$

   Geometrically, this states that in a parallelogram, the sum of the squares of the lengths of the diagonals equals the sum of the squares of the lengths of the sides.

   b) Show that a parallelogram is a rectangle if and only if its diagonals have the same length: $\|X+Y\|=\|X-Y\|$.

2. Strang p. 193 #22, 28

3. Strang p. 202-203, #2, 3, 5, 6, 7

4. Strang p. 204 #13, 16

5. a) Find an orthonormal basis for the plane $x-y-2z=0$ in $\mathbb{R}^3$.

   b) In $\mathbb{R}^3$, find the P matrix that projects a vector orthogonally into the plane $x-y-2z=0$ using *both* of the following methods:

   METHOD 1. Find an orthonormal basis for the plane and use it to construct the projection.

   METHOD 2. Find a vector **e** that is orthogonal to the plane and compute the orthogonal projection Q along this vector. Then $P=I-Q$ (why?).

   c) In $\mathbb{R}^3$ compute the distance from the plane $x-y-2z=3$ to the origin.

   d) In $\mathbb{R}^3$ compute the distance from the point $(1,1,1)$ to the plane $x-y-2z=3$.

6. Strang p. 229-230 #15, 16, 18

7. Strang p. 230 #20, 24

8. Strang p. 231 #30, 31,

9. Strang p. 231-231 #33, 34

10. If a square matrix A has the property that $A^2 = A$ (this is the general property of a projection) and the null space of A is zero, show that $A = I$.

11. Strang p. 228 #2, 3, 5

12. Let $\mathbf{n} := (a, b, c) \in \mathbb{R}^3$ be a *unit* vector and $\mathcal{S}$ the plane of vectors (through the origin) perpendicular to $\mathbf{n}$.

    a) Show that the *orthogonal projection of* $\mathbf{x}$ *in the direction of* $\mathbf{n}$ can be written in the matrix form

    $$\langle \mathbf{x}, \mathbf{n} \rangle \mathbf{n} = (\mathbf{n}\mathbf{n}^T)\mathbf{x} = \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

    where $\langle \mathbf{x}, \mathbf{n} \rangle$ is the usual inner product, $\mathbf{n}^T$ is the transpose of the column vector $\mathbf{n}$, and $\mathbf{n}\mathbf{n}^T$ is matrix multiplication.

    b) Show that the *orthogonal projection* P of a vector $\mathbf{x} \in \mathbb{R}^3$ into $\mathcal{S}$ is

    $$P\mathbf{x} = \mathbf{x} - \langle \mathbf{x}, \mathbf{n} \rangle \mathbf{n} = (I - \mathbf{n}\mathbf{n}^T)\mathbf{x},$$

    Apply this to compute the orthogonal projection of the vector $\mathbf{x} = (1, -2, 3)$ into the plane in $\mathbb{R}^3$ whose points satisfy $x - y + 2z = 0$.

    c) Find a formula similar to the previous part for the orthogonal *reflection* R of a vector across $\mathcal{S}$. Then apply it to compute the orthogonal reflection of the vector $\mathbf{v} = (1, -2, 3)$ across the plane in $\mathbb{R}^3$ whose points satisfy $x - y + 2z = 0$.

    d) Find a 3×3 matrix that projects a vector in $\mathbb{R}^3$ into the plane $x - y + 2z = 0$.

    e) Find a 3×3 matrix that reflects a vector in $\mathbb{R}^3$ orthogonally across the plane $x - y + 2z = 0$.

13. a) Let $v \in \mathbb{R}^n$ be a unit vector and Px the orthogonal projection of $x \in \mathbb{R}^n$ in the direction of v, that is, if $x = \text{const.}\, v$, then $Px = x$, while if $x \perp v$, then $Px = 0$. Show that $P = vv^T$ (here v^T is the transpose of the column vector v). In matrix notation, $(P)_{ij} = v_i v_j$.

    b) Continuing, let Q be the orthogonal projection into the subspace perpendicular to v. Show that $Q = I - vv^T$.

    c) Let u and v be orthogonal unit vectors and let R be the orthogonal projection into the subspace perpendicular to both u and v. Show that $R = I - uu^T - vv^T$.