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Newton's method for finding square roots

Let $A > 0$ be a positive real number. We want to show that there is a real number x with $x^2 = A$. We already know that for many real numbers, such as $A = 2$, there is no rational number x with this property. Formally, let $f(x) := x^2 - A$. We want to solve the equation $f(x) = 0$.
Newton gave a useful general recipe for solving equations of the form $f(x) = 0$. Applied to compute square roots, so $f(x) := x^2 - A$, it (see below) gives

$$x_{k+1} = \frac{1}{2}\left(x_k + \frac{A}{x_k}\right). \tag{1}$$

Clearly, if the initial approximation is positive, $x_1 > 0$ (we'll assume this) then all of the x_k are positive. To get some sense of these approximations, in the special case where $A = 3$ and the initial approximation is $x_1 = 1$ I used a calculator and found (to 20 decimal accuracy)

$$x_2 = 2.0, \quad x_3 = 1.75 \quad x_4 = 1.7321428571428571428$$
$$x_5 = 1.7320508100147275405 \quad x_6 = 1.7320508075688772952$$

while the exact number is $\sqrt{3} = 1.7320508075688772935$, so x_6 above is already *very* close. Beginning with x_2 the successive approximations seem to be decreasing. To investigate this we compute $x_{n+1} - x_n$. From (1), by simple algebra we find that

$$x_{k+1} - x_k = \frac{A - x_k^2}{2x_k}. \tag{2}$$

Thus, there are two cases: Case 1 is $x_k^2 > A$. Here $x_{k+1} < x_k$. Case 2 is $x_k^2 < A$. Here $x_{k+1} > x_k$.
If we are in Case 1 for x_k, are we also in Case 1 for x_{k+1}? We compute:

$$x_{k+1}^2 - A = \left(\frac{x_k^2 + A}{2x_k}\right)^2 - A = \frac{(x_k^2 - A)^2}{4x_k^2}. \tag{3}$$

Since the right hand side is always positive (lucky!), we see that beginning with $k = 2$ we are always in Case 1, no matter if we start in Case 1 or Case 2. Consequently beginning with x_2 the sequence is monotone decreasing. Because it is bounded below, the x_k converge to some limit $L > 0$. From (2) since the left side converges to zero it is clear that $A - L^2 = 0$ so $L = \sqrt{A}$.
The inequality (3) also yields a valuable estimate of the rate of convergence. This is easiest to appreciate if we look at the case where $A \geq 1$ Because $x_k^2 > A > 1$ (for $k \geq 2$) we have

$$x_{k+1}^2 - A \leq \frac{(x_k^2 - A)^2}{4A^2} \leq (x_k^2 - A)^2 \tag{4}$$

Thus at each step, the error, $x_{k+1}^2 - A$, is less than the *square* of the error in the previous step. For instance, if $x_k^2 - A < 10^{-5}$, then $x_{k+1}^2 - A < 10^{-10}$, an increase of *doubling* the number of decimal point accuracy. Now that we know $\sqrt{A}$ exists, it is easy to veryfy the related error estimate

$$x_{k+1} - \sqrt{A} = \frac{1}{2x_k}(x_k - \sqrt{A})^2. \tag{5}$$

This confirms that the rapid convergence of the numerical experiment we did at the beginning was not a coincidence.

Newton's Method is a useful general recipe for solving equations of the form $f(x) = 0$. Say we have some approximation x_k to a solution. He showed how to get a better approximation x_{k+1} . It works most of the time if your approximation is close enough to the solution. Here's the procedure. Go to the point $(x_k, f(x_k))$ and find the tangent line. Its equation is

$$y = f(x_k) + f'(x_k)(x - x_k).$$

The next approximation, x_{k+1}, is where this tangent line crosses the x axis. Thus,

$$0 = f(x_k) + f'(x_k)(x_{k+1} - x_k), \qquad \text{that is,} \qquad x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

Applied to compute square roots, so $f(x) := x^2 - A$, this gives

$$x_{k+1} = \frac{1}{2}\left(x_k + \frac{A}{x_k}\right),$$

which is what we used in (1).

[Last revised: September 4, 2018]