Zeroes of $\zeta(s)$ on $\sigma = 1$: There are none.

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Write $s = \sigma + it$ and p will always be a prime number. We will show that the *Riemann Zeta Function*

$$\zeta(s) = \sum_n^\infty \frac{1}{n^s} = \prod_{\text{primes } p} \left(1 - \frac{1}{p^s}\right)^{-1} \tag{1}$$

has no zeroes on the line $\sigma = 1$. The above factorization involving the primes was found by Euler for real s. Riemann made the observation that one gains insight by continuing $\zeta(s)$ to complex s. Note that since $|n^s| = n^\sigma$, then for any $\delta > 0$ the series for the zeta function converges uniformly in the half-plane $\operatorname{Re} s \geq 1 + \delta$ so $\zeta(s)$ is analytic in the half-plane $\operatorname{Re}\{s\} > 1$.

We follow Hadamard's original version with a simplification by Mertens. Most current expositions give a slighter shorter proof, but it then becomes too mysterious for my taste.

The first step is to continue $\zeta(s)$ analytically to a larger region.

Lemma 1 $\zeta(s) - \dfrac{1}{s-1}$ *can be continued to the half-plane Re* $\{s\} > 0$ *as a holomorphic function.*

REMARK With a bit more work Riemann even showed that $\zeta(s) - \frac{1}{s-1}$ can be continued as an entire function.

Proof of the Lemma. For $\operatorname{Re}\{s\} > 1$

$$\zeta(s) - \frac{1}{s-1} = \sum_{n=1}^\infty \frac{1}{n^s} - \int_1^\infty \frac{1}{x^s}\,dx = \sum_{n=1}^\infty \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s}\right) dx. \tag{2}$$

Although the mean value theorem is not valid for complex-valued C^1 functions $f(t)$, the *inequality*

$$|f(b) - f(a)| \leq \int_a^b |f'(t)|\,dt \leq \max_{a \leq t \leq b} |f'(t)|(|b-a|$$

is still correct. Using it with $f(t) = t^{-s}$, $n \leq t$ we obtain the estimate

$$\left|\int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s}\right) dx\right| \leq \max\left|\frac{s}{x^{s+1}}\right| \leq \frac{|s|}{n^{\operatorname{Re}\{s\}+1}}.$$

Thus for any $\delta > 0$ the infinite series on the right side of (2) converges absolutely and uniformly in the half-plane $\operatorname{Re}\{s\} \geq \delta$, so it gives an analytic continuation of the right side of (2), and hence $\zeta(s)$ to the half-plane $\operatorname{Re}\{s\} > 0$.

Theorem 2 $\zeta(s)$ *has no zeroes on the line Re* $\{s\} = 1$.

Since for $|t| < 1$ we know $-\log(1-t) = t + \dfrac{t^2}{2} + \dfrac{t^3}{3} + \dfrac{t^4}{4} + \cdots$, then

$$\log \zeta(s) = -\sum_p \log(1 - p^{-s}) = \sum_p \sum_{n=1}^\infty \frac{p^{-ns}}{n}$$

that is,

$$\zeta(s)=\exp\left[\sum_p\sum_{n=1}^{\infty}\frac{p^{-ns}}{n}\right].$$

Because $s=\sigma+it$ we have $p^{-ns}=p^{-n\sigma}p^{-nit}=p^{-n\sigma}e^{-nit\log p}$ so

$$\operatorname{Re}\{p^{-ns}\}=p^{-n\sigma}\cos(nt\theta_p),\quad \text{where}\quad \theta_p=\log p.$$

Therefore

$$|\zeta(s)|=\exp\operatorname{Re}\left\{\sum_p\sum_{n=1}^{\infty}\frac{p^{-ns}}{n}\right\}=\exp\left[\sum_p\sum_{n=1}^{\infty}\frac{p^{-n\sigma}}{n}\cos(nt\theta_p)\right].$$

The key observation is that the dependence of $|\zeta(\sigma+it)|$ on t arises only in the $\cos(nt\theta_p)$ term. Since $\cos 2x == 2\cos^2 x-1$, This gives a relationship between $\zeta(\sigma)$, $|\zeta(\sigma+it)|$, and $|\zeta(\sigma+2it)|$. To exploit this, for any integers α, β, and γ note that

$$\zeta(\sigma)^{\alpha}|\zeta(\sigma+it)|^{\beta}|\zeta(\sigma+2it)|^{\gamma}=\exp\left(\sum_p\sum_{n=1}^{\infty}\frac{p^{-n\sigma}}{n}[\alpha+\beta\cos(nt\theta_p)+\gamma\cos(2nt\theta_p)]\right). \tag{3}$$

However, writing $u=\cos x$ and using $\cos 2x == 2\cos^2 x-1$,

$$\alpha+\beta\cos x+\gamma\cos 2x=(\alpha-\gamma)+\beta\cos x+2\gamma\cos^2 x=2\gamma\left(u^2+\frac{\beta}{2\gamma}u+\frac{\alpha-\gamma}{2\gamma}\right).$$

Pick $\frac{\beta}{2\gamma}=2$ and $\frac{\alpha-\gamma}{2\gamma}=1$, then, with say $\gamma=1$, so $\alpha=3$ and $\beta=4$ (there are many other equally useful choices),

$$\alpha+\beta\cos x+\gamma\cos 2x=2(1+u)^2\geq 0.$$

Thus, for this choice the exponent in (3) is non-negative so for any $s=\sigma+it$ with $\sigma>0$.

$$\zeta(\sigma)^3|\zeta(\sigma+it)|^4|\zeta(\sigma+2it)|\geq 1. \tag{4}$$

We will use this to show that the assumption that $\zeta(1+ib)=0$ gives a contradiction. Let $\sigma=1+\delta$. Since on $\operatorname{Re}\{s\}=1$ we know $\zeta(s)$ is analytic except for a simple pole at $s=1$, then for sufficiently small $\delta>0$

$$|\zeta(1+\delta+ib)|\leq \text{const}\,\delta,\quad |\zeta(1+\delta+2ib)|\leq \text{const},\quad \text{and}\quad |\zeta(1+\delta)|\leq\frac{\text{const}}{\delta}.$$

Thus, for any real positive α, β, γ:

$$\zeta(1+\delta)^{\alpha}|\zeta(1+\delta+ib)|^{\beta}|\zeta(1+\delta+2ib)|^{\gamma}\leq \text{const}\,\delta^{-\alpha}\delta^{\beta} \tag{5}$$

so if $\beta>\alpha$, the right hand side of (5) converges to zero as $\delta\to 0$. This contradicts (4) for the particular values of $\alpha=3$, $\beta=4$, and $\gamma=1$.