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Complex Analysis Exam I

DIRECTIONS This exam has two parts, Part A has 7 short answer problems (35 points) while Part B has 5 traditional problems (65 points).
Closed book but you may use one 3×5 card with notes (on both sides).

All contour integrals are assumed to be in the positive sense (counterclockwise).

Short Answer Problems 7 problems [5 points each] (35 points total)
For A1–A5 let $f(z)$ be holomorphic for $0 < |z| < \infty$. What can you say about $f(z)$ if you are told the following? Briefly justify your assertions.

A1. $|z^2 f(z)| < 5$.

A2. $|f(z)| \to \infty$ as $|z| \to 0$.

A3. $f''(z) + f(z) = 0$ for all real rational $z \neq 0$.

A4. $|f(z)| \leq |z| + 1$ and $f(\frac{1}{n}) = 0, \quad n = 1, 2, \ldots$.

A5. $|f(z)| \leq |f(3)|$ for $|z - 3| < 2$.

A6. If $\sum_{n=0}^{\infty} a_n z^n$ represents the function $\dfrac{\sin z}{z^2 + 2}$, whch of the following are true? (Why?)

(A). Converges for $z = 1$.

(B). Converges absolutely for $z = 1$.

(C). Converges absolutely for $z = 2$.

A7. Describe the singularities of $\varphi(z) := \dfrac{1 - \cos(z^5)}{\sin^3 z}$ at $z = 0, \quad z = \pi$, and $z = \infty$.

Traditional Problems [13 points each] (65 points total)

B1. Prove the Fundamental Theorem of Algebra: that any polynomial

$$p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0, \quad n > 0,$$

with complex coefficients a_k has exactly n complex roots, counted with multiplicity.

B2. Let $f(z)$ be an entire function that is a one-to-one map from $\mathbb{C}$ to $\mathbb{C}$. Show that $f(z) = az + b$ for some complex constants $a \neq 0$ and b.

B3. Evaluate $A = \displaystyle\int_0^\infty \frac{\cos xt}{x^2 + a^2}\, dx$ where $a > 0$ and $t > 0$.

B4. Let $h(z)$, $z = x + iy$, be holomorphic in the strip $|y| < 10$ with $|h(z)| < 1$ there. Prove that $\cos z + h(z)$ has an infinite number of zeroes in this strip. [NOTE: You may use without proof that $|\cos z|^2 = \cosh^2 y - \sin^2 x$].

B5. Let a function $f(z)$ have all of the properties

(a). holomorphic in $\{|z| \leq 1\}$,

(b). $|f(z)| = 1$ for $|z| = 1$,

(c). $f(a) = 0$ for some $|a| < 1$,

(d). $f(z) \neq 0$ for all $|z| \leq 1$, with $z \neq a$.

Prove that $f(z) = \alpha \left(\dfrac{z - a}{1 - \bar{a}z} \right)^k$, where $|\alpha| = 1$ and k is a positive integer.