Complex Analysis — May 13, 1961

1. Map the region between the lines $x-y=1$, $x-y=2$ conformally onto the upper half-plane, $\operatorname{Im}\zeta \geq 0$.

2. Find a region in the $z-$plane in which the function e^{z^2} assumes every value (except zero) *exactly once*.

3. Determine how many linearly independent homogeneous polynomials $P_n(x,y)$ of degree n in two real variables x and y exist such that $\frac{\partial^2 P_n}{\partial x^2}+\frac{\partial^2 P_n}{\partial y^2}=-0$.

4. Assuming that the values of $\int_0^\infty e^{-x^2}\,dx=\frac{\sqrt{\pi}}{2}$ is known, compute

$$\int_0^\infty \cos(x^2)\,dx \quad \text{and} \quad \int_0^\infty \sin(x^2)\,dx$$

by looking at these as integrals in the complex plane. Sketch briefly the necessary convergence arguments.

5. Find the three power or Laurent series expansions for $\frac{1}{z^2-1}$ such that every point except $z=\pm 1$ is a point of absolute convergence of one of these series.

6. Let $f(z)=\frac{z}{e^z-1}=\sum_{n=0}^{\infty} b_n z^n$ be the power series expansion of $f(z)$ in a neighborhood of $z=0$. Find $\limsup_{n\to\infty}|b_n|^{1/n}$ and show that $b_{2n+1}=0$ for $n=1,2,3,\ldots$.

Complex Analysis — March 12, 1962

1. Find a function $w=f(z)$ mapping the sector $|\arg z|<\alpha<\pi$ conformally onto the unit disk $\{|w|<1\}$. Describe the behavior of $f(z)$ near $z=0$.

2. Evaluate the following integrals

$$a).\ \oint_{|z|=3}\frac{\sin(z+1)}{z(z+1)}\,dz, \qquad b).\ \oint_{|z|=3}\frac{z(z+1)}{\sin(z+1)}\,dz, \qquad c).\ \oint_{|z|=3} z^7 e^{1/z}\,dz.$$

where the integration is taken counterclockwise.

3. Let $\sum_{n=0}^\infty a_n z^n$, $\sum_{n=0}^\infty b_n z^n$ have radii of convergence r_a and r_b, respectively. What can be said about the radius of convergence of

$$a).\ \sum_{n=0}^\infty (a_n+b_n)z^n \quad \text{and} \quad b).\ \sum_{n=0}^\infty a_n b_n z^n\ ?$$

4. Let $\phi(t)$ be a continuous function of t for $0\leq t\leq 1$ and let $f(z):=\int_0^1 \frac{z\phi(t)}{t-z^2}\,dt$.
   a) For which values of z does $f(z)$ represent an analytic function?
   b) Find the Laurent expansion at infinity.

5. Let $f(z):=z+\sum_{n=2}^\infty a_n z^n$ have a positive radius of convergence.
   a) Does there exist a series $g(w)=w+\sum_{n=2}^\infty b_n w^n$ satisfying
   $$f(g(w))=w?$$
   b) Is the series $g(w)$ uniquely determined? Does it have a positive radius of convergence? Why?

6. How many roots does the equation $\frac{1}{2}e^z+z^4+1=0$ possess in the left half-plane $\operatorname{Re} z<0$? Justify your assertion.

G63.2460 Complex Variable — Final Examination — May 15, 1963 — Prof. Nirenberg

1. a) Find the harmonic function which equals $\bar{z}^5 z^2 + z^3 \bar{z}^2$ on $|z| = 1$.
   b) Find the harmonic function which equals $4x^3 - y^2$ on $|z| = 1$.

2. Let $f(z)$ be analytic and bounded in the upper half plane and continuous in its closure. Suppose that $|f| \le 1$ on the real axis. Prove $|f| \le 1$ everywhere.

3. Assume that $f_n(z)$ is a sequence of analytic functions in $|z| \le 1$ converging uniformly to $f(z) \not\equiv 0$. If $f(0) = 0$ and if w is sufficiently close to zero prove that there are positive numbers δ, N such that for all $n > N$ the equation $f_n(z) = w$ has at least one root in $|z| \le \delta$.

4. Let $f(z) \not\equiv 0$ be meromorphic in $|z| \le 1$ and let $a_1, \ldots, a_n$, be its zeros in $|z| \le 1$ and $b_1, \ldots, b_m$ its poles in $|z| \le 1$. If $f(0) \ne 0, \infty$ prove that

$$\frac{1}{2\pi}\int_0^{2\pi} \log |f(e^{i\theta})| d\theta = \log |f(0)| + \sum_{j=1}^{n} \log \frac{1}{|a_j|} - \sum_{k=1}^{m} \log \frac{1}{|b_k|} .$$

Hint: Use Blaschke product.

5. Evaluate the Fourier transform of $u(x) = \frac{x}{(1+x^2)^2}$, i.e.

$$\hat{u}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{x\, e^{-ix\xi}}{(1+x^2)^2} dx \quad \text{for all } \xi .$$

6. Let p be a polynomial of the form

$$p(z) = z^n + a_{n-1} z^{n-1} + \ldots + a_0$$

and let $f(z)$ be analytic for $|z| \le 1$. Prove that

$$|f(0)| \le \max_{|z|=1} |f(z)p(z)| .$$

Hint: Consider $\frac{f(z)p(z)}{B(z)}$ where $B(z)$ is the Blaschke product corresponding to the roots of p in $|z| < 1$.

Comprehensive Examination — **September 1962**

Complex Variables

1. Represent all complex values of $(-1)^i$, and $(1+i)^{2/3}$ in the form $a+bi$.

2. a) For each condition below give an example of a function analytic in $\{0 < |z| < 1\}$ which
   (i) has a simple pole at $z = 0$ and vanishes at $z = 1/2$.
   (ii) has an essential singularity at $z = 0$ and a pole of order 2 at $z = 1$.
   b) Map the unit circle conformally onto the half-strip $\{\mathrm{Re}\, w > 0\}$, $\{|\mathrm{Im}\, w| < \pi\}$.

3. Evaluate the following integrals – justifying your answers.
   a) $\oint \frac{e^{z^2} - e^z}{e^{2z} - 1} dx$ on $|z| = 1$.
   b) $\oint \frac{e^{z^2} - e^z}{e^{2z} - 1} dx$ on $|z| = 15$.
   c) $\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+1)} dx$

4. F is analytic in $|z| < 10$ and $\mathrm{Im}\, F = \sin\theta$ on $|z| = 1$. Find F in $|z| < 10$. Justify your answer.

5. Where does the series $\sum_{n=1}^{\infty} \frac{z^n}{n(n+1)}$ converge? Why? Express it in terms of elementary functions.

6. Prove thet there is *no* function analytic in $|z| \le 1$ such that

$$|f(z)| < 1 \quad \text{on} \quad |z| = 1, \qquad f(\tfrac{1}{2}) = 0, \quad \text{and} \quad f(-\tfrac{1}{2}) = \frac{19}{20}.$$

1. Determine the constants A and B so that the function

$$\frac{1}{z^2} + \frac{A}{(z-1)^2} + \frac{B}{z(z-1)}$$

has a zero of highest possible order at ∞.

2. Develop

$$\frac{1}{z(z+1)^2\ (z+2)^3}$$

in partial fractions.

3. Give a complete proof showing that the reduced form

$$R(z) = \frac{P(z)}{Q(z)}$$

of a rational function is unique except for a common constant factor in P and Q. (In other words, if $P/Q = P_1/Q_1$ and both fractions are reduced, show that $P_1 = cP$, $Q_1 = cQ$ with some constant c).

4. Show first that

$$R(z) = \frac{a_0 + a_1 z + \cdots + a_n z^n}{\bar{a}_n + \bar{a}_{n-1} z + \cdots + \bar{a}_0 z^n}$$

satisfies $|R(z)| = 1$ on the unit circle $|z| = 1$. Next prove that the most general rational function with that property has the above form except for a factor cz^m with $|c| = 1$. (Hint: Rember that $|z| = 1$ gives $\bar{z} = 1/z$).

5. If $\lim\limits_{n\to\infty} z_n = A$, prove that

$$\lim_{n\to\infty} \frac{1}{n}(z_1 + \cdots + z_n) = A.$$

(Suggestion: Why may one as well assume that $A = 0$?)

6. Find the radius of convergence of

$$\sum n^p z^n,\ \sum \frac{z^n}{n!},\ \sum n!\ z^n,\ \sum z^{n!}$$

7. If $f(z) = \sum a_n z^n$, what is $\sum n^3 a_n z^n$?

Math. 213a Homework Due Nov. 1. 1965

1. Find e^z for $z = -\frac{\pi i}{2}, \frac{3}{4}\pi i, \frac{2}{3}\pi i$.

2. For what values of z is e^z equal to $2, -1, i, -i/2, -1+2i$?

3. Find the real and imaginary parts of $\exp(e^z)$.

4. Determine all values of $2^i, i^i, (-1)^{2i}$.

5. Show that $|\cos z|^2 = \frac{1}{2}(\cosh 2y + \cos 2x)$ and find a corresponding expression for $|\sin z|^2$.

6. Express arc tan w in terms of the logarithm.

7. Give a definition of the "angles" in a triangle, and <u>prove</u> that the sum of the angles is π.

Math. 213a Hour-examination Dec. 6, 1965

1. Expand $\frac{z+1}{z^2(z-1)}$ in partial fractions.

2. What are the values of $(1+i)^i$?

3. For what values of z is

$$\sum_1^{\infty} n\left(\frac{1-z}{1+z}\right)^n$$

convergent, and what is the sum?

4. Prove that a continuous function from one metric space to another maps connected sets on connected sets.

5. Find the image of the region $1 < |z+1| < 2$ under the mapping $w = \frac{z^2}{z+1}$. Is the mapping one to one?

6. The circle $|z-1| = 1$ is mapped by $w = \frac{z+i}{2z-1}$. Where is the center of the image circle?

7. What is the value of

$$\int_\gamma |z|^2 dz$$

where γ is the clock-wise boundary of the first quadrant of $|z| < 1$.

8. In the following integrals C is the circle $|z| = 2$ in the positive sense. Find

a) $\int_C \frac{z\,dz}{z-1}$, b) $\int_C \frac{dz}{z^2-1}$, c) $\int_C \frac{e^z dz}{(z-1)^2}$.

9. What is

$$f(z) = \frac{1}{2\pi i}\int_C \frac{\varphi(\zeta)\,d\zeta}{\zeta - z}$$

if C is the unit circle (positive sense) and $\varphi(\zeta) = \zeta + \zeta^{-1}$. (Different answers for $|z| < 1$ and $|z| > 1$).