Math. 213a Homework Due Dec. 13, 1965

1. Compute

$$\int_{|z|=1} \frac{e^z}{z}\,dz, \qquad \int_{|z|=1} \frac{e^z}{z^n}\,dz, \qquad \int_{|z|=2} \frac{dz}{z^2+1}$$

where the circles are in the positive sense.

2. Compute

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2}$$

where a is a constant, $|a| \neq \rho$. (Hint: use the equations $z\bar{z} = \rho^2$ and $|dz| = -i\rho\, dz/z$ on the circle).

3. Find the possible values of

$$\frac{1}{2\pi i} \int_{|z|=1} \frac{dz}{(z-a)(z-b)}$$

for different positions of a and b.

In what cases can the answer be read off directly from Cauchy's integral theorem or integral formula, and in what case should one use partial fractions?

4. Find $\int_{|z|=2} z^n (1-z)^m\, dz$ for various integral values of n and m (positive or negative).

5. Prove that a function which is analytic in the whole plane and satisfies an inequality $|f(z)| < |z|^n$ for some n and all sufficiently large $|z|$ must reduce to a polynomial.

6. If $f(z)$ is analytic for $|z| < 1$, and if it is known that $|f(z)| \leq (1-|z|)^{-1}$, find the best upper bound for $|f^{(n)}(0)|$ that Cauchy's estimate will yield.

MATHEMATICS 213a
FINAL EXAMINATION
PROFESSOR AHLFORS

JANUARY 28, 1966
2:15 P.M.
JEFFERSON 250

ANSWER ALL QUESTIONS

1. Find all solutions of the equation $\sin z = i$.

2. If the series $\sum_0^\infty a_n z^n$ has radius of convergence R_1, and if $\sum_0^\infty b_n z^n$ has radius of convergence R_2, prove that the radius of convergence of $\sum_0^\infty a_n b_n z^n$ is at least $R_1 R_2$.

   Show by example that there is no upper bound.

3. Suppose that the linear transformation $w = S(z)$ maps the unit circle $|z| = 1$ onto the real axis. If $S(0) = i$, on what circle must $S(1/2)$ lie? Prove further that all points on this circle are possible values of $S(1/2)$.

4. Let D be the circular segment defined by $x > 1/2$, $|z| < 1$. Find explicitly the conformal mapping $w = f(z)$ which maps D on the unit disk in such a way that $f(1/2) = -1$ and

$$f\left(\frac{1 \pm i\sqrt{3}}{2}\right) = \pm i.$$

5. Prove Cauchy's theorem for a rectangle (Goursat's proof).

(over)

6. If C is the circle $|z| = 1$ in positive sense, evaluate

$$\int_C \frac{z^2 dz}{(2z + a)(z - a)}$$

for different values of a.

7. Let D_r be the part of the annulus $r < |z| < 1$ situated in the first quadrant.

   a) Define z^i as a single-valued function in D_r.

   b) Find the smallest r such that the mapping $w = z^i$ is one to one in D_r.

   c) For this value of r, describe the image region. Is it simply connected?

8. Use the calculus of residues to compute

   a) $$\int_0^\infty \frac{dx}{(x^2 + a^2)(x^2 + b^2)} \qquad (a, b > 0)$$

   b) $$\int_0^\infty \frac{\cos xt}{x^2 + a^2}\, dx \qquad (a \text{ and } t \text{ real})$$

1. Find all the residues of $\cot^3 z$.

2. Recall that $f(z)$ is regular or has a pole at ∞ if $f_1(z) = f(1/z)$ is regular or has a pole at 0. When this is so, show that one can write

$$f(z) = B_m z^m + \cdots + B_1 z + B_0 + \frac{B_{-1}}{z} + g(z)$$

where $g(z)$ has at least a double zero at ∞.

3. In the preceding situation, define the residue of $f(z)$ at ∞ as $-B_{-1}$. With this definition, if $f(z)$ is analytic except for poles outside a closed curve C, prove that

$$\frac{1}{2\pi i}\int_C f(z)dz = -\sum \text{Res } f$$

where the sum is over all residues outside of C, including the one at ∞ (C is counterclockwise).

Corollary: The sum of <u>all</u> residues of a rational function is zero.

4. Use the preceding exercise to find at a glance the values of

$$\int_C \frac{z^2 dz}{2z^4 + 1}, \quad C \text{ is } |z| = 1$$

$$\int_C \frac{z^3 dz}{2z^4 + 1}, \quad C \text{ is } |z| = 1$$

$$\int_C \frac{dz}{(z-3)(z^5 - 1)}, \quad C \text{ is } |z| = 2.$$

5. Given that $\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$ (Math. 105b) find

$$\int_0^\infty \cos x^2\, dx \quad \text{and} \quad \int_0^\infty \sin x^2\, dx.$$

(Integrate along the boundary of a sector)

Math. 213 b Homework page 2

6. Evaluate the Cauchy principal value of

$$\int_0^\infty \frac{x^{p-1}dx}{1-x} \qquad (0 < p < 1)$$

7. If $\rho < 1$, prove that for sufficiently large n the polynomial

$$P_n(z) = 1 + 2z + 3z^2 + \cdots + nz^{n-1}$$

has no zeros in $|z| < \rho$.

Math. 213b Homework Due Feb. 28, 1966

1. Use residues to compute

$$\iint_{|z|<1} \frac{dx\,dy}{|z-a|^2}$$

where $|a| > 1$.

2. If $f(z)$ is analytic for $|z| \leq 1$, find the value of

$$\frac{1}{2\pi}\int_0^{2\pi} \overline{f(e^{i\theta})}\ \frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta.$$

3. Use the preceding result for another proof of Schwarz' formula

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\,\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta \qquad (f = u + iv).$$

4. If $U(t)$ is piecewise continuous and bounded for all real t, show that

$$P_U(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2}\,U(t)\,dt$$

represents a harmonic function in the upper half plane with boundary values $U(t)$ (at points of continuity).

5. A special case of the preceding formula is

$$\omega(z) = \int_a^b \frac{y}{(x-t)^2+y^2}\,dt.$$

Show that $\omega(z)$ represents the angle under which the segment (a, b) is seen from z. Discuss the behavior of $\omega(z)$ as $z \longrightarrow a$ or b.

6. Find the geometric meaning of

$$\omega(z) = \int_\alpha^\beta = \frac{1-|z|^2}{|e^{i\theta}-z|^2}\,d\theta.$$

What are the level curves $\omega(z) = \text{const}$.

(25)

Mathematics 246 Name____________________

Functions of a Complex Variable

Final Exam (Friday Section)

May 20, 1960, 6:00 P.M. to 8:00 P.M.

Please write on these sheets.

I. Definitions. Complete the following:

a. The function $f(z)$ has a pole of order p at the (finite) point $z = z_0$ if

b. The function $f(z)$ has a zero of order p at $z = z_0$ if

c. The principle part of $f(z)$ at a pole at $z = z_0$ is

d. The residue of $f(z)$ at a pole at $z = z_0$ is

e. If $f_1(z)$ and $f_2(z)$ are holomorphic in domains D_1 and D_2, respectively, $f_2(z)$ is a direct analytic continuation of $f_1(z)$ if

f. If $f_1(z)$ and $f_2(z)$ are holomorphic in domain D_1 and D_2, respectively, $f_2(z)$ is an analytic continuation of $f_1(z)$ if

g. A simple closed curve is a natural boundary of $f(z)$ if

h. A mapping $z \to w = f(z)$ is one-to-one in a domain D of the z-plane if

i. A smooth mapping $z \to w = f(z)$ is conformal at $z = z_0$ if

j. A function is meromorphic in a domain D if

k. A function is meromorphic in the extended complex plane if

II. General Theory.

a. If $f(z)$ and $g(z)$ are holomorphic in a domain D and $f(z) = g(z)$ at a sequence of distinct points $z = z_1, z_2, \ldots\ldots,$ when can you conclude that $f(z) = g(z)$ for all z in D?

b. If $f(z)$ is an entire function such that $|f(z)| < 1$ for $|z| > 10^{10}$, what can you conclude?

c. If $z \to w = f(z)$ is a one-to-one conformal mapping of a Jordan domain D_1 onto a domain D_2, what additional information can you give to make the mapping unique?

d. If Γ_1 and Γ_2 are two curves going from point α to point β, when can you be sure that analytic continuation

along Γ_1 will give the same result as along Γ_2?

III. Special questions.

a. Find the expansion, in powers of z, of the function

$$f(z) = \frac{1}{2\pi i} \int_C \frac{e^{(t - z^2/t)}}{t} \, dt \quad ,$$

where C is the unit circle $|t| = 1$, described counter-clockwise.

b. Identify the functions defined by the following infinite products. (Hint: note where the zeros are.)

$$z \prod_1^\infty (m) \left(1 - \frac{z^2}{m^2\pi^2}\right) =$$

$$\prod_0^\infty (m) \left(1 - \frac{z^2}{(m+\frac{1}{2})^2\pi^2}\right) =$$

$$z \prod_1^\infty (m) \left(1 + \frac{z^2}{m^2\pi^2}\right) =$$

c. Evaluate the following integrals by the residue theorem. (Describe the contour used.)

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1} =$$

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+1} =$$