Math. 213b Homework Due March 21

1. Find an explicit expression for

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^2 + a^2}$$

2. Prove that

$$(1+z)(1+z^2)(1+z^4)(1+z^8)\ldots = \frac{1}{1-z}$$

for $|z| < 1$.

3. Prove Gauss' formula

$$(2\pi)^{\frac{n-1}{2}}\,\Gamma(z) = n^{z-\frac{1}{2}}\,\Gamma\left(\frac{z}{n}\right)\,\Gamma\left(\frac{z+1}{n}\right)\cdots\Gamma\left(\frac{z+n-1}{n}\right).$$

4. Find the residue of $\Gamma(z)$ at $z = -n$.

5. Use Jensen's formula to eval uate

$$\int_0^{2\pi} \log|\sin(\rho e^{i\theta})|\, d\theta.$$

Mathematics 213

Final Examination May 27, 1960

Professor Mackey 2:15 p.m.

1. Prove that the absolute value of a non-constant analytic function cannot take on a maximum in a connected open set.

2. Use the theory of residues to evaluate

$$\int_0^{\infty} \frac{dx}{1+x^6}$$

3. Define elliptic function, primitive period, order of an elliptic function, Weierstrass $\wp$ function . Prove that the $\wp$ function satisfies a differential equation of the form $\left(\frac{d\wp}{dz}\right)^2 = Q(\wp)$ where Q is a polynomial. What does this imply about the relationship between elliptic functions and elliptic integrals?

4. State without proof the main facts about the convergence of infinite products leading up to and including the Weierstrass theorem about the existence of entire functions with given zeros. What can you conclude from the Weierstrass theorem about the existence of meromorphic functions? State a theorem (Mittag Leffler's) about the existence of meromorphic functions in the plane in which the zeros of the function play no role.

5. State the Riemann mapping theorem. Give an outline of the proof in which you make clear the nature of each main step but omit details.

6. Define (abstract) Riemann surface, continuation of a power series, complete analytic function. Without going into all the technical details of the existence proof, explain what is meant by the Riemann surface of a complete analytic function.

(OVER)

7. Let f be an entire function which is bounded in the open wedge-shaped region S bounded by two intersecting half lines and let $M = \text{l.u.b.}_{z \in S} |f(z)|$. Assuming that $f'(z) \neq 0$ for any $z \in S$ and that the wedge angle is an irrational multiple of a straight angle, show that there exists W with $|W| < M$ such that $f(z) = W$ implies $z \notin S$.

8. Let O_1 and O_2 be open subsets of the complex plane and let ϕ be a conformal map of O_1 on O_2. Let g be meromorphic on O_2 and let $f = g \circ \phi$. Let Z_o be a pole of f. Show that the residue of g at $\phi(Z_o)$ need not be the same as that of f at Z_o. Then show how preservation of residues in conformal mapping can be obtained by associating them with "differentials" instead of functions.

EXAM

Prof. L. Bers
Math 246 sec A

March 28, 1962

Answer 4 questions:

1. Find a Möbius transformation which maps the circle $|z| = 1$ onto the circle $|w| = 1$ and the circle $|z - \frac{3}{10}| = \frac{3}{10}$ onto a circle $|w| = r < 1$. Find r.

2. Map the region $z = re^{i\theta}$, $0 \leq r \leq 1$, $0 \leq \theta \leq \pi/2$ conformally onto the upper half-plane so that the points 0, 1, i are taken into the points 0, 1, ∞, respectively.

3. Find all conformal mappings of the domain exterior to the two circles $|z - 1| = 1$, $|z + 1| = 1$ (and containing ∞) onto the upper half-plane.

4. Prove the following theorem. Let D be a simply connected domain, $f(z)$ a holomorphic function defined in D with $f(z) \neq 0$ in D. Then there exist holomorphic functions $g(z)$, $g_j(z)$, $j = 1, 2, \ldots$ in D such that

$$f(z) = e^{g(z)} = g_j(z)^j$$

5. Prove the following theorem. If $\{f_n(z)\}$ is a sequence of holomorphic functions defined in $|z| < 1$, if $\text{Im}\, f_n(z) \geq 0$ and $f_n(0) = i$, then the sequence $\{f_n\}$ contains a normally convergent subsequence.

MATHEMATICS 213

Dr. H. Rohrl

Monday, January 21, 1963
9:15 a.m.

(1) Define for topological spaces both compactness and connectedness. Give necessary and sufficient conditions for subsets of the plane to be compact (for open subsets of the plane to be connected).

(2) Let A be an open subset of the plane and m a meromorphic function in A which has only finitely many poles in A. Show that m is the quotient of two functions both of which are holomorphic in A.

(3) Let f(t) be a continuous function of the real variable t where t is contained in the closed interval [a, b]. Prove that

$$F(z) = \int_a^b \frac{f(t)}{t - z}\,dt \qquad , z \in \mathbb{C} - [a, b]$$

is a holomorphic function. What can you say about boundary values of F(z) as z approaches a point of the open interval (a, b)?

(4) Let f be holomorphic in the disk $D = \{z : |z| < a\}$. Show that $|f(0)| < \inf \{|f(z)| : |z| = a\}$ implies the existence of at least one zero of f in D. Using this result show that every non constant polynomial has at least one zero.

(5) Let $P_1, \ldots, P_n$ be points of the plane and A an open subset of the plane. Denoting the distance of the points P and P_r by $\overline{PP}_r$, prove that for positive real numbers $a_1, \ldots, a_n$ the functions

OVER

$$f_1(P) = \overline{PP}_1 \cdot \ldots \cdot \overline{PP}_n \qquad , P \in \overline{A}$$

$$f_2(P) = a_1\overline{PP}_1^{\,2} + \ldots + a_n\overline{PP}_n^{\,2} \qquad , P \in \overline{A}$$

assume their maximum on the boundary of A.

(6) Suppose that $f(z) = \sum \{a_n z^n : n = 0, 1, \ldots\}$ is holomorphic for $|z| \leq 1$. Show that among all polynomials $P_N(z)$ of fixed degree N there is one and only one for which

$$\frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\varphi}) - P_N(e^{i\varphi})|^2 d\varphi$$

assumes its minimum. Determine this polynomial.

(7) Let f_n be a sequence of holomorphic functions in the disk $D = \{z : |z| < 1\}$ which is uniformly bounded. Assume that none of the functions f_n has a zero in D and that $\lim_{n \to \infty} f_n(0) = 0$. Prove the the sequence f_n converges in D to the function 0.

Mathematics 213

Professor Ahlfors
Final Examination

May 29, 1962
9:15 a.m.

1. Prove Riemann's mapping theorem.

2. Suppose that $f(z)$ has a pole of order k at 0 . Find the singular part of the Laurent development of

$$f''' / f' - \frac{3}{2} (f''/f')^2$$

at the origin.

3. The functions $\zeta(z)$ and $\sigma(z)$ are obtained from Weierstrass' p-function by the conditions $\zeta'(z) = -p(z)$ and $\sigma'(z)/\sigma(z) = \zeta(z)$, together with customary normalizations.

   a) Show that $\eta_k = \zeta(z + \omega_k) - \zeta(z)$ is constant $(k = 1,2)$.

   b) Prove that $\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i$.

   c) Derive formulas for $\sigma(z+\omega_k)$, $k = 1,2$.

4. Let f be analytic in a simply connected region D , and suppose that all zeros of f have even multiplicity. Prove that there exists a singlevalued analytic function g in D which satisfies $g(z)^2 = f(z)$. (Hint: use the monodromy theorem.)

5. F is an entire function, and $F_n(z) = F(nz)$, $n = 1,2,\dots$. Prove that the functions F_n form a normal family in the annulus $1 < |z| < 2$ if and only if F is a polynomial.

(OVER)

6. What is a harmonic function? Prove the mean-value property and the maximum principle. Outline the reasoning which shows that a continuous function with the mean-value property is harmonic.

7. Let t_0, t_1, t_2, t_3, t_4 be distinct real numbers. Describe the shape of the region on which

$$w = \int_{t_0}^{z} \sqrt{\frac{z-t_0}{(z-t_1)(z-t_2)(z-t_3)(z-t_4)}}\, dz$$

maps the upper halfplane. What happens if $t_0 = t_1$ or $t_1 = t_2$, the others remaining distinct?

(END)

MATHEMATICS 213a

Professor Walsh

January 23, 1965
9:15 a.m.

1. What is a complex number? If α and β are complex numbers, define $\alpha\beta$ and prove $\alpha\beta = \beta\alpha$.

2. Define <u>degree</u> of a rational function. Show that a rational function of z takes on (in the extended plane) every value a number of times equal to the degree of the rational function if the function is not a constant.

3. Define <u>analytic function</u>. Show that the integral of a function analytic in a rectangle over the perimeter of the rectangle is zero.

4. If $f(z)$ is analytic interior to a disk, state and establish Cauchy's integral formula for a closed curve in the disk and a point in the disk not on the curve.

5. State and prove the principle of maximum modulus.

6. Define for an analytic function: removable singularity, pole, essential singularity. Establish the fundamental property of an essential singularity of such a function relative to limits approached.

7. State and prove Rouché's theorem.

OVER

8. Evaluate

$$\int_0^\infty \frac{\sin^2 kx}{x^2} \, , \; k > 0 \, .$$

9. Prove that if each function $f_n(z)$ is analytic in a region Ω , and if the sequence $\{f_n(z)\}$ converges uniformly to the limit $f(z)$ in Ω , then $f(z)$ is analytic in Ω . Indicate where the uniformity of convergence is used in the proof.

10. A function $f(z)$ is continuous in the region $|z| < 1$, and analytic there except when z is real. Show that $f(z)$ is analytic throughout the region.