MATHEMATICS 213

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Final Examination
Professor Mackey

February 3, 1960
9:15 a.m.

1. Find all of the fifth roots of $1 + \sqrt{3}i$ and all values of $(-1)^{1+i}$.

2. Assuming the least upper bound property of the real numbers, prove that any Cauchy sequence of complex numbers converges to some complex number. You may not assume the Cauchy sequence property of the real numbers. Prove it if you need it.

3. For subsets of the plane define compactness, boundedness, closedness, openedness and connectedness. Illustrate each definition with an example and a counter-example.

4. Define analytic function and give as many properties as you can think of which are equivalent to analyticity.

5. Let $a_0 + a_1 z + a_2 z^2 + \dots$ be a power series which converges to

$$\frac{z^3 - 1}{(z^2 + z + 1)(z^2 + 3z - 4)}$$

for all $z$ with $|z|$ sufficiently small. Compute the largest possible $\epsilon$ such that the series in question converges for $|z| < \epsilon$. What can you say about the series if $|z| > \epsilon$.

6. Starting from Cauchy's integral formula for circles (which you may assume to be true) prove that every non-constant polynomial has at least one root.

7. Define isolated singularity and describe in two ways the three way classification into removable singularities, poles, and essential singularities. Define residue and find the residue of $\cos z/(z^2 + 1)$ at $z = i$.

8. Outline the main points in the reading period assignment up to and including the formulation of the general form of the Cauchy integral theorem.

FINAL EXAMINATION

Math. 609
Prof. Kazdan

Due: Wed. Dec. 16, 2:00 P.M.

Directions:
1. Answer as many problems as you can - but at least 6 problems. More are needed for an A.
2. Be neat and accurate.
3. Delete any straightforward computations.
4. You may not work jointly.
5. You may use any notes or references. If a result is done in some reference, do not recopy it, but merely give the page and name of the reference.
6. Papers handed in after 2:00 P.M. will not be accepted.
7. Graded papers may be picked up in my office between 1-2 P.M. on Thurs., Dec. 17. If you prefer, you may give me a self-addressed envelope.
8. PLEASE DO THE PROBLEMS IN THE ORDER LISTED ON THIS SHEET.

1. (a) Prove that the roots of the polynomial

$$p(z) = z^n + a_{n-1}z^{n-1} + \dots + a_0$$

depend continuously on the coefficients, $\{a_k\}$.

(b) If $f$ is analytic in a neighborhood of $z_0$ and if $f$ has a zero of order $n > 1$ at $z_0$, prove that $g(z) = f(z) - a$ has $n$ simple zeroes in a neighborhood of $z_0$ for all sufficiently small complex numbers $a$.

A PAUSE FOR SOME NOTATION:

$K$ = open unit disk, centered at 0.

$K_r$ = open disk of radius $r < 1$, centered at 0.

$U$, $V$ = simply connected open sets in $\mathbb{C}$.

2. Let $f$ be analytic in $\bar{U} \subset V$. If $\lim_{z \to \partial U} |f(z)| =$ constant, prove that

$$f(z) \frac{h(z) - a_j}{1 - \bar{a}_j h(z)}$$

for some $h$ analytic some constants $a_j$, $c$.

3. Let $f$ be analytic in $|f(z)| \leq M$ in $K$ and $f(0) \neq 0$. Prove that

(number of zeIn $K_r$) $\leq \frac{1}{\log r} \log \frac{|f(0)|}{M}$.

(Hint: consider $g(z)$)

4. Find the following con: $f: U \to K$, where $U$ is:

(a) intersection of {1 $\{|z - 1| < 1\}$,

(b) region outside of $1\} \cup \{|z + 1| \leq 1\}$,

(c) $K \cap \{x > \frac{1}{2}\}$.

5. Suppose the linear tra $w = S(z)$ maps the unit circle onto the rith $S(0) = i$.

(a) On what circle muIe?

(b) Prove that every nis circle is a possible value of $S(\frac{1}{2})$.

6. Let $f$ be an isomorphI to U and let $U_r = f(K_r)$.

(a) If $h$ is an auter $U$ leaving $f(0)$ fixed, prove that $U_r$. (Hint: apply the Schwarz Lemma f ).

(b) If $U$ is convex, $U_r$ is also convex. (Hint: let $|z_1|$ and consider $g(z;t) = z_2) + tf(z)$ ).

(c) More generally, i open disk in $\bar{K}$, prove that $f(E)$ is con Hint: Prove there is an



automorphism $q$ of $K$ such that $q(E) = K_r$, and consider $f \circ q^{-1}$)

7. Let $f$ and $F$ be analytic in $K$ with $F$ univalent there, and let $U = f(K)$, $V = F(K)$. If $f(0) = F(0)$ and $U \subset V$, prove that there is a function $g$ analytic in $K$ such that

$f = F \circ g$, with $|g(z)| \leq |z|$.

Moreover, show that $f(K_r) \subset F(K_r)$.

8. Look up the definition of "natural boundary" for an analytic function. Prove that for any connected open set $D$ in the complex plane, there is a function $f$ analytic in $D$ having $D$ as a natural boundary. (Remark: if you can not do the general case, then do what you can).

9. Let $f$ be an entire analytic function with the properties:

(a) $f(x + 2\pi) = f(x)$ for any real $x$,

(b) $|f(z)| \leq \exp a|z|$, for all $z$ and some $a > 0$.

Prove that $f$ has the form

$$f(z) = \sum_{k=-n}^{n} a_k e^{ikz}, \text{ where } n \leq a.$$

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MATHEMATICS 221

Final Examination

January 22, 1960
9:15 a.m.

1. Define normal family of functions. Show that a family of functions analytic and bounded in a region is normal there.

2. State and prove the Riemann mapping theorem for a simply connected region.

3. Let the functions $f_n(z)$ analytic and schlicht in $|z| < 1$ converge almost uniformly there to $f_o(z)$, with $f_n(0) = 0$, and let K be a continuum containing $w = 0$ which lies in the image of $|z| < 1$ under the transformation $w = f_o(z)$. Show that for n sufficiently large K lies in the image of $|z| < 1$ under the transformation $w = f_n(z)$.

4. State and prove the Bieberbach Flächensatz.

5. If the region D of the z-plane is star-shaped (with respect to the origin) and if $w = f(z)$ maps D onto $|w| < 1$, then the subregion $|f(z)| < r$, $0 < r < 1$, of D is also star-shaped.

6. Show that a doubly connected region bounded by two disjoint Jordan curves can be mapped conformally onto an annulus.

7. Show by methods of potential theory that any simply connected region possessing a Green's function can be mapped conformally onto a circle.

8. Show that an arbitrary smooth surface which is topologically equivalent to a sphere can be mapped conformally onto a sphere.

9. State the Uniformization Theorem and outline its proof.

Professor J. L. Walsh

(84)

Math. 213a Homework Due Nov. 22, 1965

1. Find the linear transformation which carries 0, i, -i into 1, -1, 0.

2. Show that any four distinct points can be carried by a linear transformation into positions 1, -1, k, -k where k depends on the points. How many solutions?

3. Find the linear transformation which carries $|z| = 2$ into $|z + 1| = 1$, the point -2 into the origin, and the origin into i.

4. What is the most general transformation of $|z| < 1$ onto itself? of the upper half plane onto itself?

5. Find a linear transformation which carries $|z| = 1$ and $|z - \frac{1}{4}| = \frac{1}{4}$ into concentric circles. What is the ratio of the radii?

6. Find the fixed points of
$w = \frac{z}{2z-1}$, $w = \frac{2z}{3z-1}$, $w = \frac{3z-4}{z-1}$, $w = \frac{z}{2-z}$.
Which of these transformations are elliptic, hyperbolic, or parabolic?

7. Find all circles which are orthogonal to $|z| = 1$ and $|z-1| = 4$.

Math. 213 a    Homework    Due November 29, 1965

In these exercises all mappings are to be conformal and you are expected to give an explicit expression for the analytic function which yields the required mapping.

1. Map the common part of the disks $|z| < 1$ and $|z - 1| < 1$ onto the inside of the unit circle. Choose the mapping so that the two symmetri[es] are preserved.

2. Map the region between $|z| = 1$ and $|z - \frac{1}{2}| = \frac{1}{2}$ on a half plane.

3. Map the complement of the arc $|z| = 1$, $y \geq 0$ on the outside of the unit circle so that the points at $\infty$ correspond to each other.

4. Map the outside of the parabola $y^2 = 2px$ on the disk $|w| > 1$ so that $x = 0$ and $x = -\frac{p}{2}$ correspond to $w = 1$ and $w = 0$.

5. Map the outside of the ellipse $(x/a)^2 + (y/b)^2 = 1$ onto $|w| < 1$ with preservation of symmetries.

6. Map the part of the z-plane to the left of the right-hand branch of the hyperbola $x^2 - y^2 = 1$ on a half plane.
(Hint: Consider on one side the mapping of the upper half of the region by $w = z^2$, on the other side the mapping of a quadrant by $w = z^3 - 3z$).



Math. 213a    Hour-examination    Dec. 6, 1965

1. Expand $\frac{z + 1}{z^2(z - 1)}$ in partial fractions.

2. What are the values of $(1 + i)^i$?

3. For what values of $z$ is
$$\sum_1^\infty n\left(\frac{1 - z}{1 + z}\right)^n$$
convergent, and what is the sum?

4. Prove that a continuous function from one metric space to another maps connected sets on connected sets.

5. Find the image of the region $1 < |z + 1| < 2$ under the mapping $w = \frac{z^2}{z + 1}$. Is the mapping one to one?

6. The circle $|z - 1| = 1$ is mapped by $w = \frac{z + i}{2z - 1}$. Where is the center of the image circle?

7. What is the value of
$$\int_\gamma |z|^2 dz$$
where $\gamma$ is the clock-wise boundary of the first quadrant of $|z| < 1$.

8. In the following integrals $C$ is the circle $|z| = 2$ in the positive sense. Find

a) $\int_C \frac{z\,dz}{z - 1}$, b) $\int_C \frac{dz}{z^2 - 1}$, c) $\int_C \frac{e^z dz}{(z - 1)^2}$.

9. What is
$$f(z) = \frac{1}{2\pi i} \int_C \frac{\varphi(\zeta)\,d\zeta}{\zeta - z}$$
if $C$ is the unit circle (positive sense) and $\varphi(\zeta) = \zeta + \zeta^{-1}$.
(Different answers for $|z| < 1$ and $|z| > 1$).

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Math. 213b Homework — Due February 7, 1966

1. If $f(z)$ is analytic in $|z| \leq 1$ and if $|f(z)| = 1$ when $|z| = 1$, show by use of the reflection principle that $f(z)$ is a rational function.

2. Prove that

$$\lim_{n \to \infty} \left(1 + \frac{z}{n}\right)^n = e^z$$

uniformly on any compact set. (Use the series expansion of $\log(1 + \frac{z}{n})$).

3. Show that

$$\varphi(s) = \sum_{n=1}^{\infty} n^{-s}$$

converges for $\mathrm{Re}\, s > 1$, that it represents an anyaltic function (known as Riemann's zeta-function) and that $\varphi'(s)$ can be obtained by term-wise differentiation.

4. Prove that

$$(1 - 2^{1-s})\,\varphi(s) = 1^{-s} - 2^{-s} + 3^{-s} - \cdots$$

when $\mathrm{Re}\, s > 1$, and that this series converges for $\mathrm{Re}\, s > 0$. (Suggestion: estimate $n^{-s} - (n+1)^{-s}$).

5. If $\frac{1}{1+z^2}$ is developed in powers of $z - a$ where $a$ is a real number, what is the radius of convergence. Find the development. (Suggestion: use partial fractions).

6. Develop $\log\left(\frac{\sin z}{z}\right)$ in powers of $z$ up to the term $z^6$.

7. Find the first three non-zero terms in the development of $\tan z$ by dividing the sine and the cosine series. (Check with the book, 2nd. ed. p. 182, 1st. ed. p. 146).

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Math 213b — Homework — Due March 14, 1966

1. The expression

$$\{f, z\} = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

is known as the Schwarzian derivative of $f$. If $f$ has a pole or zero of order $m$ at $z_0$, find the leading term in the Laurent development of $\{f, z\}$.

2. Find the Taylor series of $(\log(1 - z))^2$ about the origin. (Give the general expression for the n th coefficient).

3. Find the Laurent series of $\cot z$ about the origin up to the term $z^3$.

4. Compare the development in the preceding exercise with what you can get from the partial fraction development of $\cot z$. Show that you can thus find the values of $\sum_1^{\infty} \frac{1}{n^2}$ and $\sum_1^{\infty} \frac{1}{n^4}$.

5. What is the canonical product development of $\cos(\sqrt{z})$, and what is its genus?

6. Show that if $f(z)$ is of genus $0$ or $1$ with only real zeros, and if $f(z)$ is real for real $z$, then all the zeros of $f'(z)$ are real. Hint: use the canonical product and consider $\mathrm{Im}\, f'(z)/f(z)$.

# MATHEMATICS 213b

Professor Walsh — June 5, 1965, 9:15 a.m.

1. Prove the validity of the Cauchy-Hadamard formula for the radius of convergence of a power series.

2. It is sometimes stated that a power series converges uniformly in its circle of convergence. Correct this statement and prove the intended theorem.

3. A function with period $2\pi i$ is analytic at every finite point of the plane. Derive a formula for the function.

4. State, and outline the proof of, the Riemann mapping theorem.

5. Prove that any two elliptic functions with the same periods are connected by an algebraic relation.

6. State and prove a theorem expressing the mean value property as sufficient for the harmonicity of a function.

7. Show that a function harmonic and bounded for all $z$ is identically constant.



OVER



8. Define <u>subharmonic function</u>. Show that a continuous function $v(z)$ is subharmonic in a region $\Omega$ if and only if we have

$$v(z_0) \leqq \frac{1}{2\pi}\int_0^{2\pi} (v(z_0 + re^{i\theta})d\theta$$

for every disk $|z - z_0| \leqq r$ in $\Omega$.

9. Show how an arbitrary region of finite connectivity can be mapped onto an annulus minus a number of circular arcs.

10. Without the use of the Riemann mapping theorem, and by study of the level loci of Green's function $G(z,0)$ for a simply connected region $\Omega$ (having at least two boundary points and containing the origin) with pole in the origin, where $G(z,0) \longrightarrow -\infty$ as $z \longrightarrow 0$, show that the function $w = \exp[G(z,0) + iH(z)]$ maps $\Omega$ one-to-one and conformally onto $|w| < 1$. Here $H(z)$ indicates a function conjugate to $G(z,0)$ in $\Omega$.

Final Examinati	May 27, 1963
Professor Walsh	2:15 p.m.

1. State and ıarz's Lemma. State explicitly any form of the of Maximum Modulus that you use.

2. State and Bieberbach Area Theorem (Flächensatz).

3. Show that ıction $z + a_2 z^2 + \ldots$ is analytic and schlicopen unit disk, then $|a_2| \leqq 2$ .

4. Show that ·n unit disk is mapped conformally onto a convex r·h circle in the disk whose center is the origin onto a convex curve.

5. Let the fu $_1(z)$, $f_2(z), \ldots,$ analytic and schlicht in a simpl d region D , with $f_1(0) = 0$ and 0 in D , continuously there to a function $f(z)$ not identi tant. Show that if a closed bounded set E is in the map of D by $f(z)$ , then E is containmap of D by each $f_n(z)$ for n sufficient

6. Outline th the Riemann mapping theorem.

7. Show how a functions arise naturally in the conformal map ltiply connected regions.

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8. Show that a (suitably smooth) surface which is topologically equivalent to a sphere can be mapped conformally onto a sphere.

9. State the Uniformization Theorem and outline its proof.