Math 313 | Exam 1 | Jerry L. Kazdan
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DIRECTIONS This exam has three parts, the first is short answer (*36 points*), the second is multiple choice (*15 points*), and the third has traditional problems (*50 points*). Closed book, no calculators – but you may use one 3" × 5" card with notes.

Part A: Short answer (6 problems, 6 points each)

A–1. Let A be a matrix, not necessarily square. Say $\mathbf{V}$ and $\mathbf{W}$ are particular solutions of the equations $A\mathbf{V} = \mathbf{Y}_1$ and $A\mathbf{W} = \mathbf{Y}_2$, respectively, while $\mathbf{Z} \neq 0$ is a solution of the homogeneous equation $A\mathbf{Z} = 0$. Answer the following in terms of $\mathbf{V}$, $\mathbf{W}$, and $\mathbf{Z}$.

a) Find some solution of $A\mathbf{X} = 3\mathbf{Y}_1 - 5\mathbf{Y}_2$.

b) Find *another* solutions of $A\mathbf{X} = 3\mathbf{Y}_1 - 5\mathbf{Y}_2$.

A–2. Let A be an invertible matrix. If $AV = 3V$ for some vector V, compute $A^{-2}V$.

A–3. In $\mathbb{R}^n$, if $X = U + V$ where U and V are orthogonal vectors, show that $\|X\| \geq \|U\|$.

A–4. Find a 2×2 (real) matrix B ($B \neq \pm I$) with the property $B^4 = I$.

A–5. Find the (orthogonal) projection of $\mathbf{x} := (1, 2, 0)$ into the plane spanned by the orthogonal vectors $\mathbf{u} := (1, 0, 1)$ and $\mathbf{v} := (1, 1, -1)$.

A–6. You are asked to fit some data $(x_1, y_1), \ldots, (x_n, y_n)$ to a curve of the form $y = \dfrac{a}{b + x}$, where the constants a and b are to be found. Transform this curve so that the method of least squares can be applied.

Part B: Multiple-choice questions (5 questions, 3 points each).
For each situation below, **circle** *all* of the possibilities that can occur:

A) no solution B) unique solution C) infinitely many solutions

Example: For this example the assertions B) and C) can occur but not A).

A (B) (C) homogeneous system $Ax = 0$ of 2 equations in 2 unknowns

1) A B C inhomogeneous system $Ax = b$ of 7 equations in 7 unknowns

2) A B C inhomogeneous system $Ax = b$ of 4 equations in 7 unknowns

3) A B C homogeneous system $Ax = 0$ of 4 equations in 7 unknowns

4) A B C inhomogeneous system $Ax = b$ of 5 equations in 2 unknowns

5) A B C homogeneous system $Ax = 0$ of 5 equations in 2 unknowns

Part C: Traditional Problems (5 problems, 10 points each)

C–1. Consider the system of equations

$$\begin{aligned} x + y - \ z &= a \\ x - y + 2z &= b \\ 3x + y \quad &= c \end{aligned}$$

a) Find the general solution of the homogeneous equations.

b) If $a = 1$, $b = 2$, and $c = 4$, then a particular solution of the inhomogeneous equations is $x = 1$, $y = 1$, $z = 1$. Find the most general solution of these inhomogeneous equations.

C–2. Let A be a matrix representing a linear map from $\mathbb{R}^n \to \mathbb{R}^k$. Show that

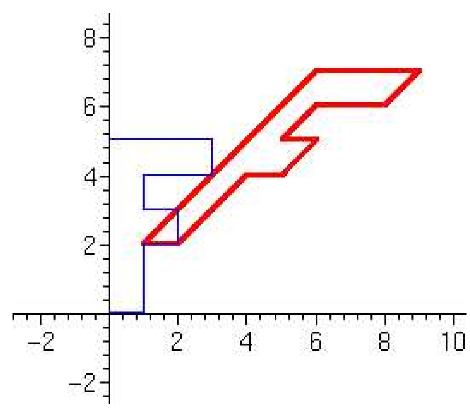
$$\dim \mathcal{N}(A) - \dim \mathcal{N}(A^T) = n - k. \qquad \text{[Here } \mathcal{N}(A) \text{ is the nullspace of } A\text{].}$$

C–3. Note that the columns vectors in the following equations are orthogonal. Use this observation to solve the equations (*no credit for any other method*).

$$\begin{aligned} x + y + z + w &= 2 \\ x + y - z - w &= 3 \\ x - y + z - w &= 0 \\ x - y - z + w &= -5 \end{aligned}$$

C–4. a) A linear map $A : \mathbb{R}^3 \to \mathbb{R}^3$ first rotates the yz-plane by $+90°$ (leaving the x-axis fixed), followed by a reflection across the xy-plane. Find the matrix representation for A in the standard basis for $\mathbb{R}^3$.

b) Find the vector V and the matrix A so that the map $F(X) = V + AX$ does the mapping indicated in the figure:



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C–5. In an experiment you measure the temperature Temp (t) every six hours on a winter day and get the data:

t	0	6	12	18
Temp(t)	2	4	8	6

Say you suspect this data should be periodic every 24 hours and have the special form

$$\text{Temp}\,(t) = a + b\sin\left(\frac{2\pi t}{24}\right) + c\cos\left(\frac{2\pi t}{24}\right).$$

a). Write the (over determined) system of equations you would like to solve ideally for a, b, and c.

b). Use the method of least squares to write the *normal equations* for the coefficients a, b, c.

c). Explicitly solve the equations you found in part b).