Math 313/515, Spring 2005 Jerry L. Kazdan

Homework Set 10, Due Thursday, April 7, 2005
(*Late papers will be accepted until 4 PM on Fri. April 8*)

1. If $A = \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}$, compute A^2, A^3, and e^A.

2. Solve the differential equation

$$u'' + 2u' + 2u = 0 \quad \text{with} \quad u(0) = 1,\ u'(0) = 0$$

by rewriting it as a first order system and using a matrix technique.

3. [Strang, p. 318 #24] Let A be a square matrix.
   a) Show that A is invertible ("non-singular") if and only if $\lambda = 0$ is not an eigenvalue.
   b) Give two reasons why e^A is never singular:
      (i). Write down its inverse.
      (ii). Write down its eigenvalues: If $Ax = \lambda x$, then $e^A x = \underline{\quad} x$.

4. [Strang, p. 380 #1] In class we worked with the linear space of polynomials $\mathcal{P}_3$ pf polynomials $p(x)$ of degree at most 3 (its dimension is 4).

   The derivative, $D := d/dx$ dedines a linear map $D : \mathcal{P}_3 \to \mathcal{P}_3$ by the usual rule $Dp = dp/dx$.
   a) Using the standard basis $p_i = 1,\ p_2 = x,\ p_3 = x^2,\ p_4 = x^3$ for $\mathcal{P}_3$, compute Dp_1, Dp_2, Dp_3, Dp_4 in terms of this basis.
   b) Write a 4×4 matrix M representing this linear map D.
   c) Compute the matrix M^2 and show that it represents the second derivative map $D^2 : \mathcal{P}_3 \to \mathcal{P}_3$.
   d) What do you think $M^4 = ?$. Explain your reasoning.

5. [Strang, p. 303 #41] Say Q is a given 5×5 matrix. We want to find a 5×5 matrix A that solves the equation $AQ - QA = 0$. This is 25 equations for 25 unknowns (the elements of A). Show there is always a solution A other than the trivial solution $A = 0$. [There are many ways to do this].

The next three problems involve Markov Chains. These were discussed in the guest lecture by Prof. Bleher and are treated in Chapter 8.3 of the Strang text.

6. A behavioral psychologist places a rat each day in a cage with two doors, A and B. The rat can go through door A, where it receives a mild electric shock, or to door B, where it receives food. A record is made of the door through which the rat passes. At the start of the experiment, on a Monday, the rat is equally likely to pass through door A as through door B. After going through door A, and receiving a shock, the probability of going through the same door on the next day is 0.3. After going through door B, and receiving food, the probability of going through the same door on the next day is 0.6.

   a) Write down the transition matrix for the Markov process.

   b) What is the probability of the rat going through door A on Wednesday (the second day after starting the experiment)?

   c) What is the steady-state vector?

7. There is a long long queue to buy tickets for a Modest Mouse concert. Everyone sees the person selling tickets whisper to the first in line that "Yes, there are still some tickets left." She tells the guy behind her and so on down the line.

   However, Modest Mouse fans are not reliable transmitters. If one is told "yes", there is only an 80% chance that person will report "yes" to the next person. On the other hand, being optimistic, if one hears "no", there is a 40% chance that fan will report "yes". If the queue is very long, what fraction of them will hear "there are no tickets left"?

8. There are two local branches of the Limousine Rental Company, one at the Airport and one in the City, as well as branches Elsewhere.

   Say every week of the limousines rented from the Airport 25% are returned to the City and 2% to branches located Elsewhere. Similarly of the limousines rented from the City 25% are returned to Airport and 2% to Elsewhere. Finally, say 10% of the limousines rented from Elsewhere are returned to the Airport and 10% to the City.

   a) If initially there are 35 limousines at the Airport, 35 in the City, and 150 Elsewhere, what is the long-term distribution of the limousines?

   b) Say M is the Markov transition matrix for this problem. Use Matlab to compute M^3, M^{10}, and M^{40}.