Math 313/515, Spring 2005 Jerry L. Kazdan

Homework Set 7, Due Thursday, March 17, 2005

(*Late papers will be accepted until 4 PM on Fri. March 18*)

1. a) Compute (by hand) the determinant of the following matrix:

$$A := \begin{pmatrix} 1 & 2 & -1 & 0 \\ -1 & -2 & 3 & 1 \\ 0 & -1 & 4 & -3 \\ 2 & 5 & 0 & 1 \end{pmatrix}$$

   by reducing it to the determinant of a lower triangular matrix.

   b) Compute the volume of the 4-dimensional "parallelogram" Q spanned by the vectors $V_1 = (1,-1,0,2)$, $V_2 = (2,-2,-1,5)$, $V_3 = (-1,3,4,0)$, $V_4 = (0,1,-3,1)$.

   c) The matrix

$$B := \begin{pmatrix} 2 & 2 & 3 & -1 \\ 0 & -1 & -2 & 7 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

   can be thought of as a map of $\mathbb{R}^4$ to itself. Compute (by hand) the volume of $B(Q)$ (this is the image of Q under the map B). Of course you can use any formulas, just not a computer or a fancy hand calculator.

2. Compute (by hand) the determinant of the following matrix.

$$C := \begin{pmatrix} a & 1 & 0 & 0 & 0 \\ b & 1 & 0 & 0 & 0 \\ c & 0 & 0 & 1 & -b \\ c & 0 & 0 & 1 & -a \\ d & e & 1 & f & g \end{pmatrix}$$

3. Use Cramer's rule to solve the equation $AX = Y$, where A is given below and Y is the *column* vector $Y = (y_1, y_2, y_3)$. Then observe you have computed A^{-1}, so exhibit it.

$$A := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 4 & 9 & 1 \end{pmatrix}.$$

4. a) For what value(s) of the constant α are the vectors

$$X_1 = (1, -2, 1), \quad X_2 = (2, \alpha, 2), \quad X_3 = (1, 2, 3)$$

   linearly dependent?

   b) For what value(s) of β does the system of equations

$$\begin{aligned} x + 2y + z &= 0 \\ -2x + \beta y + 2z &= 0 \\ x + 2y + 3z &= 0 \end{aligned}$$

   have more than the trivial solution $x = y = z = 0$? (Explain your answer.)

5. Let $\Delta_n = \det M_n$ be the determinant of an $n \times n$ matrix M_n with a's along the main diagonal and b's on the two "off diagonals" directly above and below the main diagonal (this is a simple example of a *tridiagonal* matrix). Thus

$$M_5 = \begin{pmatrix} a & b & 0 & 0 & 0 \\ b & a & b & 0 & 0 \\ 0 & b & a & b & 0 \\ 0 & 0 & b & a & b \\ 0 & 0 & 0 & b & a \end{pmatrix}.$$

   a) Prove $\Delta_n = a\Delta_{n-1} - b^2\Delta_{n-2}$.

   b) Compute Δ_1 and Δ_2 by hand. Then use the formula to compute Δ_3 and Δ_4.

   REMARK: In a few weeks we will show how one can use part a). above to get a general formula for Δ_n].

6. After you write the following Matlab functions, please print a copy as part of the homework. IN ADDITION, please send a copy as an email attachment sent to our TA: `elber@math.upenn.edu` so she can test it with some real input.

   a) Given an angle `theta` (in degrees), write a Matlab function `R = rotz(theta)` that returns a 3×3 matrix corresponding to a rotation in $\mathbb{R}^3$ by `theta` degrees around the z axis.

   b) Write a Matlab function `g = Transform(A,V)` that takes a 3x3 matrix A and a vector V viewed as a 3×1 matrix and returns a 4×4 matrix, `g = [A V; 0 0 0 1]`. [This is the *homogeneous version* of the transformation generated by A and V as in Homework Set 2 #11].

END HOMEWORK SET 7

The following is for your information only. These are not assigned.

These problems outline of the solution to the **Bonus Problem** from Homework 6. First, we repeat the problem.

Bonus Problem Given a unit vector $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ in $\mathbb{R}^3$, find an explicit formula for a 3×3 matrix M that rotates a vector $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ in $\mathbb{R}^3$, about the "$\mathbf{n}$ axis" through a specified angle θ. Thus, your matrix M will involve a, b, c, and θ.

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1. a) *For this problem you should review the* cross product *of vectors in* $\mathbb{R}^3$.
   Let $\mathbf{v} := (\alpha, \beta, \gamma)$ and $\mathbf{x} := (x, y, z)$ be any vectors in $\mathbb{R}^3$. Viewed as column vectors, find a 3×3 matrix $A_{\mathbf{v}}$ so that the *cross product* $\mathbf{v} \times \mathbf{x} = A_{\mathbf{v}}\mathbf{x}$.
   ANSWER:
   $$\mathbf{v} \times \mathbf{x} = A_{\mathbf{v}}\mathbf{x} = \begin{pmatrix} 0 & -\gamma & \beta \\ \gamma & 0 & -\alpha \\ -\beta & \alpha & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$
   where the anti-symmetric matrix $A_{\mathbf{v}}$ is defined by the above formula.

   b) From this, one has $\mathbf{v} \times (\mathbf{v} \times \mathbf{x}) = A_{\mathbf{v}}(\mathbf{v} \times \mathbf{x}) = A_{\mathbf{v}}^2\mathbf{x}$ (why?). Combined with the cross product identity $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \langle \mathbf{u}, \mathbf{w} \rangle \mathbf{v} - \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{w}$, show that
   $$A_{\mathbf{v}}^2\mathbf{x} = \langle \mathbf{v}, \mathbf{x} \rangle \mathbf{v} - \|\mathbf{v}\|^2\mathbf{x}.$$

   c) If $\mathbf{n} = (a, b, c)$ is a unit vector, use this formula to show that (somewhat surprisingly) the orthogonal projection of $\mathbf{x}$ into the plane perpendicular to $\mathbf{n}$ is given by
   $$\mathbf{x} - (\mathbf{x} \cdot \mathbf{n})\mathbf{n} = -A_{\mathbf{n}}^2\mathbf{x} = -\begin{pmatrix} -b^2 - c^2 & ab & ac \\ ab & -a^2 - c^2 & bc \\ ac & bc & -a^2 - b^2 \end{pmatrix}$$
   [Compare this with HW 5 #12a-b]].

2. Recall [see HW 5 #12b] that $\mathbf{u} := \mathbf{x} - (\mathbf{x} \cdot \mathbf{n})\mathbf{n}$ is the projection of $\mathbf{x}$ into the plane perpendicular to the unit vector $\mathbf{n}$. Show that in $\mathbb{R}^3$ the vector
   $$\mathbf{w} := \mathbf{n} \times \mathbf{u} = \mathbf{n} \times [\mathbf{x} - (\mathbf{x} \cdot \mathbf{n})\mathbf{n}] = \mathbf{n} \times \mathbf{x}$$
   is perpendicular to both $\mathbf{n}$ and $\mathbf{u}$, and that $\mathbf{w}$ has the same length as $\mathbf{u}$. Thus $\mathbf{n}$, $\mathbf{u}$, and $\mathbf{w}$ are orthogonal with $\mathbf{u}$, and $\mathbf{w}$ in the plane perpendicular to the axis $\mathbf{n}$.

3. Let $\mathbf{n} \in \mathbb{R}^3$ be a unit vector. Find a formula for the 3×3 matrix that determines a rotation of $\mathbb{R}^3$ through an angle θ with $\mathbf{n}$ as axis of rotation (assuming the axis passes through the origin). Here we outline how to find this formula — but before reading further, try finding it on your own.

   a) (Example) Find a matrix that rotates $\mathbb{R}^3$ through the angle θ using the vector $(1,0,0)$ as the axis of rotation.

   b) More generally, let $\mathbf{u}$ and $\mathbf{w}$ be orthonormal vectors in the plane perpendicular to $\mathbf{n}$. Show that the map

   $$R_{\mathbf{n}} : \mathbf{x} \mapsto (\mathbf{x} \cdot \mathbf{n})\mathbf{n} + \cos\theta\,\mathbf{u} + \sin\theta\,\mathbf{w}$$

   rotates $\mathbf{x}$ through an angle θ with $\mathbf{n}$ as axis of rotation. [Note: one needs more information to be able to distinguish between θ and $-\theta$].

   c) Using problem 2 to write $\mathbf{u}$ and $\mathbf{w}$, in terms of $\mathbf{n}$ and $\mathbf{x}$, show that one can rewrite the above formula as

   $$\begin{aligned} R_{\mathbf{n}}\mathbf{x} &= (\mathbf{x} \cdot \mathbf{n})\mathbf{n} + \cos\theta\,[\mathbf{x} - (\mathbf{x} \cdot \mathbf{n})\mathbf{n}] + \sin\theta\,(\mathbf{n} \times \mathbf{x}) \\ &= \mathbf{x} + \sin\theta\,(\mathbf{n} \times \mathbf{x}) + (1 - \cos\theta)[(\mathbf{x} \cdot \mathbf{n})\mathbf{n} - \mathbf{x}]. \end{aligned}$$

   Thus, using problem 1, if $\mathbf{n} = (a,b,c) \in \mathbb{R}^3$ deduce that:

   $$R_{\mathbf{n}} = I + \sin\theta \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} + (1 - \cos\theta) \begin{pmatrix} -b^2 - c^2 & ab & ac \\ ab & -a^2 - c^2 & bc \\ ac & bc & -a^2 - b^2 \end{pmatrix}.$$

   d) Let $A_{\mathbf{n}}$ be as in Problem 1 (but using $\mathbf{n}$ rather than $\mathbf{v}$). Show that

   $$R_{\mathbf{n}} = I + \sin\theta A_{\mathbf{n}} + (1 - \cos\theta) A_{\mathbf{n}}^2.$$

   e) Use this formula to find the matrix that rotates $\mathbb{R}^3$ through an angle of θ using as axis the line through the origin and the point $(1,1,1)$.