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Math 425   Jerry L. Kazdan

Exam 2

April 26, 2011   12:00 – 1:20

DIRECTIONS This exam has three parts, Part A, short answer, has 1 problem (10 points). Part B has 4 shorter problems (9 points each, so 36 points). Part C has 3 traditional problems (15 points each so 45 points). Total is 91 points.
Closed book, no calculators or computers– but you may use one $3'' \times 5''$ card with notes on both sides.

Part A: Short Answer (1 problem, 10 points).

1. Let S and T be linear spaces and $A : S \to T$ be a linear map. Say $\mathbf{V}$ and $\mathbf{W}$ are particular solutions of the equations $A\mathbf{V} = \mathbf{Y}_1$ and $A\mathbf{W} = \mathbf{Y}_2$, respectively, while $\mathbf{Z} \neq 0$ is a solution of the homogeneous equation $A\mathbf{Z} = 0$.

   Answer the following in terms of $\mathbf{V}$, $\mathbf{W}$, and $\mathbf{Z}$.

   a) Find some solution of $A\mathbf{X} = 3\mathbf{Y}_1$. SOLUTION: $\mathbf{X} = 3\mathbf{V}$

   b) Find some solution of $A\mathbf{X} = -5\mathbf{Y}_2$. SOLUTION: $\mathbf{X} = -5\mathbf{W}$

   c) Find some solution of $A\mathbf{X} = 3\mathbf{Y}_1 - 5\mathbf{Y}_2$. SOLUTION: $\mathbf{X} = 3\mathbf{V} - 5\mathbf{W}$

   d) Find another solution (other than $\mathbf{Z}$ and 0) of the homogeneous equation $A\mathbf{X} = 0$. SOLUTION: $\mathbf{X} = 2\mathbf{Z}$

   e) Find another solution of $A\mathbf{X} = 3\mathbf{Y}_1 - 5\mathbf{Y}_2$. SOLUTION: $\mathbf{X} = 3\mathbf{V} - 5\mathbf{Y}_2 + \mathbf{Z}$

Part B: Short Problems (4 problems, 9 points each so 36 points)

B–1. Suppose f is a function of one variable that has a continuous second derivative. Show that for any constants a and b, the function

$$u(x, y) = f(ax + by)$$

is a solution of the nonlinear PDE

$$u_{xx}u_{yy} - u_{xy}^2 = 0.$$

SOLUTION: By the chain rule, $u_x = f'(ax + by)a$, so $u_{xx} = f''(ax + by)a^2$. Similarly, $u_{yy} = f''(ax + by)b^2$ and $u_{xy} = f''(ax + by)ab$. Thus,

$$u_{xx}u_{yy} - u_{xy}^2 = f''(ax + by)^2[a^2b^2 - (ab)^2] = 0.$$

B–2. $\mathbf{U} = (1,1,0,1)$ and $\mathbf{V} = (-1,2,0,-1)$ are orthogonal vectors in R^4.

Write the vector $\mathbf{X} = (1,1,1,0)$ in the form

$$\mathbf{X} = a\mathbf{U} + b\mathbf{V} + \mathbf{W}, \tag{1}$$

where a, b are scalars and $\mathbf{W}$ is a vector perpendicular to $\mathbf{U}$ and $\mathbf{V}$.

SOLUTION: Take the inner product of (1) with $\mathbf{U}$ and use that we want $\mathbf{W}$ to be orthogonal to $\mathbf{U}$ to find

$$\langle \mathbf{X},\, \mathbf{U}\rangle = \mathbf{a}\langle \mathbf{U},\, \mathbf{U}\rangle \qquad \text{so} \quad a = \frac{2}{3}.$$

Similarly,

$$\langle \mathbf{X},\, \mathbf{V}\rangle = \mathbf{b}\langle \mathbf{V},\, \mathbf{V}\rangle \qquad \text{so} \quad b = \frac{1}{6}$$

Thus,

$$X = \frac{2}{3}\mathbf{U} + \frac{1}{6}\mathbf{V} + \mathbf{W},$$

where $\mathbf{W}$ is defined by this equation. It is orthogonal to both $\mathbf{U}$ and $\mathbf{V}$ since that is how we computed a and b.

B–3. If $u(x,y)$ is a solution of the Laplace equation in the unit disk $x^2+y^2<1$ with boundary conditions

$$u(x,y) = \begin{cases} 1 & \text{for } x^2+y^2=1, \quad y>0 \\ 0 & \text{for } x^2+y^2=1, \quad y\le 0. \end{cases}$$

Compute $u(0,0)$.

SOLUTION: By the mean value property, the value of a harmonic function at the center of a disk is the average of its values on the circumference. Thus $u(0,0) = \dfrac{1}{2}$.

As an alternate, one can use the Poisson formula for the solution of the Dirichlet problem for the disk. The solution at the center (where $r=0$) is equally speedy.

B–4. This problem concerns the solution of the initial-value problem for the wave equation $u_{tt} = u_{xx} + u_{yy}$ in two space variables $(x,y) \in \mathbb{R}^2$, together with the initial conditions

$$u(x,y,0) = f(x,y), \qquad u_t(x,y,0) = 0.$$

If $f(x,y)$ is a 2π periodic functions of x, so $f(x+2\pi,y) = f(x,y)$ for all x, show that $u(x,y,t)$ is also a 2π periodic function of x.

SOLUTION: Let $v(x,y,t) = u(x+2\pi,y,t)$. Since the wave equation has constant coefficients, v also satisfies the wave equation for $(x,y)\in\mathbb{R}^2$. Because f is 2π periodic in x, v satisfies the same initial conditions. Since the solution of this initial balue problem for the wave equation is unique, $v(x,y,t) = u(x,y,t)$, which is what we wanted to show.

Part C: Traditional Problems (3 problems, 15 points each so 45 points)

C–1. Let $\Omega \subset \mathbb{R}^2$ be a bounded region in the plane.

a) Let $w(x, y, t)$ be a solution of the modified heat equation

$$w_t = w_{xx} + w_{yy} - 7w_x + w_y - 5w \tag{2}$$

for $(x, y) \in \Omega$ and $0 < t \le T < \infty$. Show that the solution w cannot have a local positive maximum or negative minimum at a point of Ω.

NOTE: There are two cases, one where the maximum point accurs at a point (x, y, t) with $0 < t < T$ and one at a point (x, y, T)

SOLUTION: If there is a positive maximum at a point (x, y, t) where $(x, y) \in \Omega$ and $0 < t < T$, then $w_x = 0$, $w_y = 0$, and $w_t = 0$, and also $w_{xx} \le 0$, $w_{yy} \le 0$ – as well as $w > 0$, This is incompatible with (2).

At a point (x, y, T), where $(x, y) \in \Omega$ the equality $w_t = 0$ is replaced by the inequality $w_t \ge 0$, but gives the same conclusion.

At at negative minimum the same reasoning applies (just replace u by $-u$).

b) If $w(x, y, 0) = \sin(x + 2y)$ for $(x, y) \in \Omega$ and $-2 \le w(x, y, t) \le 3$ for $(x, y) \in \partial\Omega$, $t \ge 0$, what can you conclude about the size of $w(x, y, t)$ for $(x, y) \in \Omega$, $t \ge 0$?.

SOLUTION: By the maximum principle, $-2 \le w(x, y, t) \le 3$ for $(x, y) \in \Omega$, $t \ge 0$.

C–2. In a bounded region $\Omega \subset \mathbb{R}^n$, let $u(x, t)$ satisfy the modified heat equation

$$u_t - 2tu = \Delta u, \tag{3}$$

as well as the initial and boundary conditions

$$u(x, 0) = f(x), \quad \text{in } \Omega \quad \text{with } u(x, t) = 0 \text{ for } x \in \partial\Omega, \quad t \ge 0. \tag{4}$$

Let $u(x, t) = \varphi(t)v(x, t)$. Show that by picking the function $\varphi(t)$ cleverly, v satisfies the standard heat equation $v_t = \Delta v$ as well as the initial and boundary conditions (4).

REMARK: This generalized to $u_t + a(t)u = \Delta u$ where $a(t)$ is any continuous function.

SOLUTION: Since $u_t = \varphi_t v + \varphi v_t$ and $\Delta u = \varphi \Delta u$, Then $u_t - 2tu = \Delta u$ becomes

$$\varphi v_t + (\varphi_t - 2t\varphi)v = \varphi \Delta v.$$

Thus pick $\varphi(t)$ so that $\varphi_t - 2t\varphi = 0$. This is a stndard ODE. It's solution is $\varphi(t) = Ce^{t^2}$ for some constant C, say $C = 1$.

It is then obvious that v has the desired properties.

C–3. The motion $u(x,y,t)$ of a special drum $\Omega \in \mathbb{R}^2$ satisfile the modified wave equation

$$u_{tt} + b(x,y,t)u_t = \Delta u \quad \text{for} \quad (x,y) \in \Omega, \qquad t > 0. \tag{5}$$

with boundary condition

$$u(x,y,t) = 0 \quad \text{for} \quad (x,y) \in \partial\Omega, \ t \geq 0. \tag{6}$$

Define the "energy"

$$E(t) := \frac{1}{2} \iint_\Omega \left[u_t^2 + |\nabla u|^2\right] \, dx\, dy.$$

Assume that $|b(x,y,t)| \leq m$ for some constant m and all $(x,y) \in \Omega, \quad t \geq 0$.

a) Show that $\quad \dfrac{dE}{dt} \leq 2mE \quad$ for all $t \geq 0$.

Solution: By Green's First,

$$\begin{aligned}
\frac{dE}{dt} &= \iint_\Omega [u_t u_{tt} + \nabla u \cdot \nabla u_t] \, dx\, dy = \iint_\Omega [u_t u_{tt} - u_t \Delta u] \, dx\, dy \\
&= \iint_\Omega u_t[-b(x,y,t)u_t] \, dx\, dy \\
&\leq m \iint_\Omega u_t^2 \, dx\, dy \leq 2mE(t).
\end{aligned}$$

b) Deduce that $\quad \dfrac{d}{dt}\left[e^{-2mt}E(t)\right] \leq 0 \quad$ for all $t \geq 0$, and hence that

$$E(t) \leq e^{2mt}E(0) \quad \text{for all} \quad t \geq 0.$$

Solution:

$$\frac{d}{dt}\left[e^{-2mt}E(t)\right] = e^{-2mt}[E' - 2mE] \leq 0,$$

so $e^{-2mt}E(t)$ is a non-increasing function fot $t \geq 0$. Thus $e^{-2mt}E(t) \leq E(0)$ for all $t \geq 0$.

c) If $u(x,y,0) = 0$ and $u_t(x,y,0) = 0$ for $(x,y) \in \Omega$, what does this say about $E(t)$ for $t \geq 0$ and hence about $u(x,y,t)$ for $t \geq 0$?

Solution: Under these assumptions $E(0) = 0$. Thus $0 \leq E(t) \leq 0$ for all $t \geq 0$, so $E(t) \equiv 0$. Therefore $u(x,y,t) = \text{const}$. But $u(x,y,0) = 0$ so $u(x,y,t) \equiv 0$ for all $t \geq 0$.