Linear ODE's

Second order linear equations

Many traditional problems involving ordinary equations arise as second order *linear* equations

$$au'' + bu' + cu = f, \quad \text{more briefly as} \quad Lu = f.$$

The problem is, given f, find u; often we will want to find u that satisfies some auxiliary initial or boundary conditions.

Here we have used the notation

$$Lu := a(x)u'' + b(x)u' + c(x)u, \tag{1}$$

so L takes a function u and gives a new function Lu. This operator L is a *linear map* because it has the two properties

$$L(\alpha u) = \alpha Lu \quad \text{and} \quad L(u+v) = Lu + Lv,$$

where α is any constant and u and v are functions. One consequence is that if $Lu = 0$ and $Lv = 0$, then $L(Au + Bv) = 0$ for any constants A and B. The solutiond of $Lu = 0$ are often called the *nullspace* or *kernel* of L. These properties show that *the nullspace of a linear map is a linear space*. For instance, in the special case where $Lu = u'' + u$ we know that $L\cos x = 0$ and $L\sin x = 0$. Thus $L(A\cos x + B\sin x) = 0$ for any constants A and B.

EXAMPLE We'll show that *any* solution of $Lu := u'' + u = 0$ has the form

$$u(x) = A\cos x + B\sin x.$$

This will show that the nullspace of $Lu := u'' + u = 0$ has *dimension two*.

First we must pick the constants A and B. Letting $x = 0$ we see that (if this is to work) $A = u(0)$. Similarly, taking the derivative we get $B = u'(0)$. Let $v(x) := u(0)\cos x + u'(0)\sin x$. Our task is to show that $u(x) = v(x)$. Equivalently, if we let $w(x) := u(x) - v(x)$, we must show that $w(x) \equiv 0$. A key observation motivating us is that by linearity, $w'' + w = 0$, and $w(0) = 0$, $w'(0) = 0$.

Introduce the function

$$E(x) = \frac{1}{2}[w'^2 + w^2].$$

Then since $w'' = -w$,

$$E'(x) = w'w'' + ww' = w'(-w) + ww' = 0,$$

so $E(x) = \text{constant}$ (in many physical examples, this is *conservation of energy*). But from $w(0) = 0$ and $w'(0) = 0$ we find $E(0) = 0$. Since $E(x)$ is a sum of squares, the only possibility is $w(x) \equiv 0$, as claimed.

This Example generalizes. Assuming $a(x) \neq 0$, the nullspace of (1) always has dimension 2. Let $\varphi(x)$ and $\psi(x)$ be the solutions of the homogeneous equation $Lu = 0$ with $\varphi(0) = 1$, $\varphi'(0) = 0$, and $\psi(0) = 0$, $\psi'(0) = 1$, then every solution of the homogeneous equation $Lu = 0$ has the form $u(x) = A\varphi(x) + B\psi(x)$ for some constants A and B. The proof, which we don't give, has two parts. The first is the *existence* of the solutions φ and ψ, the second is their uniqueness. While these proofs are not obvious, they are not killers.

For (1) – and other linear ordinary and partial differential equations, it is surprising that if one knows the general solution of the homogeneous equation $Lu = 0$ one can find an *explicit* formula for a particular solution of the inhomogeneous equation $Lu = f$.
Based on our experience with the first order linear inhomogeneous equation $u' + au = f$ it is plausable to seek u in the form $u(x) = p(x)v(x)$ where $p(x)$ is chosen cleverly to make the equation for v simple to solve. We do this for $Lu := u'' + u = f$ (the general case of (1) is then routine). Clearly

$$u' = pv' + p'v \quad \text{and} \quad u'' = pv'' + 2p'v' + p''v$$

so

$$Lu = pv'' + 2p'v' + (p'' + p)v.$$

This clearly simplifies if we pick p as a solution of the homogeneous equation $p'' + p = 0$. But we know *two* solutions of this, $\cos x$ and $\sin x$. Which should we use? After some experimentation, Lagrange decided he should use *both* and instead sought u in the more general form

$$u = pv + qw, \tag{2}$$

where for our example $p(x) = \cos x$ and $q(x) = \sin x$. Now he had one equation, $f = Lu = pv'' + 2p'v' + qw'' + 2q'w'$ for the two unknowns, $v(x)$ and $w(x)$ so he could impose another condition. After some experimenting he imposed the condition

$$pv' + qw' = 0, \tag{3}$$

which resulted in the two linear equations:

$$f = Lu = p'v' + q'w' \quad \text{and} \quad 0 = pv' + qw',$$

that is,

$$f = (-\sin x)v' + (\cos x)w' \quad \text{and} \quad 0 = (\cos x)v' + (\sin x)w'.$$

He solved these for v' and w':

$$v'(x) = -\sin x\, f(x) \quad \text{and} w'(x) = \cos x\, f(x).$$

Thus integrating and using (2), we obtain the simple formula for a particular solution, u_{part}, of the inhomogeneous equation $Lu = f$:

$$u_{\text{part}}(x) = \cos x \int_0^x -\sin s\, f(s)\, ds + \sin x \int_0^x \cos s\, f(s)\, ds = \int_0^x \sin(x-s)\, f(s)\, ds \tag{4}$$

To get the *general solution of the inhomogeneous equation* we simply add the general solution of the homogeneous equation:

$$u(x) = A\cos x + B\sin x + \int_0^x \sin(x-s)\, f(s)\, ds.$$

In honor of George Green we often write (4) in the symbolic form

$$u(x) = \int_0^x G(x,s)\, f(s)\, ds \tag{5}$$

and call $G(x,s) := \sin(x-s)$ *Green's function* for the equation $Lu = f$. The point is that (5) can be thought of as writing $u = L^{-1}f$ so we have a conceptually satisfying formula for the inverse operator L^{-1}.

Lagrange's procedure for finding the formula (4) for a particular solution of the inhomogeneous equation is called *variation of parameters*. The key step is to seek u in the form (2) with $Lp = 0$ and $Lq = 0$.
We now carry out the details for the general case of the general second order equation

$$Lu := u'' + b(x)u' + c(x)u = f(x). \tag{6}$$

Note that here the coefficient of u'' is 1 (if not, then divide by it).
Using equation (2) *and* the condition (3) he found that

$$u' = p'v + q'w, \quad \text{and} \quad u'' = p'v' + q'w' + p''v + q''w.$$

Substitute these into the equation (6) for L. After a short computation that uses $Lp = Lq = 0$, we get the simple formula

$$Lu = p'v' + q'w'. \tag{7}$$

To solve $Lu = f$ we thus need to find v and w that satisfy this and (3):

$$\begin{aligned} pv' + qw' &= 0 \\ p'v' + q'w' &= f. \end{aligned}$$

These are two linear equations for v' and w'. Their solution is

$$v' = \frac{-qf}{W} \quad \text{and} \quad w' = \frac{pf}{W},$$

where $W(x) := pq' - p'q$ (called the *Wronskian* of p and q). Integrating we find v and w – and thus from (2), a particular solution u_{part}

$$u_{\text{part}} = p(x)\int_0^x \frac{-q(s)f(s)}{W(s)}\,ds + q(x)\int_0^x \frac{p(s)f(s)}{W(s)}\,ds = \int_0^x G(x,s)f(s)\,ds, \tag{8}$$

where

$$G(x,s) := \frac{q(x)p(s) - p(x)q(s)}{W(s)}$$

is *Green's function* for the problem. In the special case of $u'' + u = f$ done earlier, $p(x) = \cos x$ and $q(x) = \sin x$ so $W(x) = 1$ and $g(x,s) = \sin(x-s)$, just as in (4).

First order linear systems

Next consider the first order system of equations

$$LU := U'(x) + A(x)U(x) = F(x), \tag{9}$$

where U and F are vectors with n components and $A(x)$ is an $n \times n$ matrix. We assume that both A and F depend continuously on x.

A typical problem is to seek a solution of (9) that satisfies some *initial condition* $U(0) = C$, where $C \in \mathbb{R}^n$ is a given vector. If $A(x)$ and $F(x)$ are both periodic with period P, another typical problem is to seek a periodic solution $U(x)$ [the simplest scalar example $u' = 1$ has no periodic solutions – with any period – and shows that answering this question may involve some work].

The homogeneous equation

A general theorem, which we'll not prove (it is not a killer) is

Theorem 1 . *Given any constant* $C \in \mathbb{R}^n$*, the homogeneous equation* $LU = 0$ *has a unique solution satisfying* $U(0) = C$.

Note that "has a unique solution" means the same as "has one and only one solution".

Let $e_1 := (1, 0, \ldots, 0)$, $e_2 := (0, 1, 0, \ldots, 0)$, ..., $e_n := (0, 0, \ldots, 0, 1)$ be the standard basis vectors in $\mathbb{R}^n$. It is useful to use the special solutions $\Phi_1(x), \ldots, \Phi_n(x)$ that satisfy the homogeneous equation $L\Phi_j(x) = 0$ with $\Phi_j(0) = e_j$ and use them to construct the $n \times n$ matrix $\Phi(x)$ whose columns are the vectors $\Phi_1(x), \ldots, \Phi_n(x)$. Then Φ satisfies

$$\Phi'(x) + A(x)\Phi(x) = 0, \quad \text{and the initial condition} \quad \Phi(0) = I.$$

This matrix $\Phi(x)$ is sometimes called the *fundamental solution matrix*.

EXAMPLE 1 Let

$$U(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}, \qquad A := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \text{and} \qquad F(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix},$$

so the system of equations $LU := U' + AU = F$ is

$$\begin{aligned} u_1' - u_2 &= f_1 \\ u_2' + u_1 &= f_2. \end{aligned} \tag{10}$$

The vectors

$$\Phi_1(x) := \begin{pmatrix} \cos x \\ -\sin x \end{pmatrix}, \qquad \Phi_2(x) := \begin{pmatrix} \sin x \\ \cos x \end{pmatrix}$$

both satisfy the homogeneous equation $L\Phi_j = 0$ with initial conditions $\Phi_1(0) = e_1$, $\Phi_2(0) = e_2$, so the fundamental solution matrix is

$$\Phi(x) := \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} \tag{11}$$

The inhomogeneous equation

Next we show that if you know a fundamental matrix solution $\Phi(x)$ for the homogeneous equation, then you can find a formula for a particular solution of the inhomogeneous equation $LU = F$, that is, $U' + AU = F$. As before, seek U in the special form $U(x) = S(x)V(x)$, where $S(x)$ is an $n \times n$ matrix. The goal is to choose a clever S so the resulting differential equation for $V(x)$ is simple. Clearly

$$LU = SV' + (S' + AS)V.$$

This evidently simplifies dramatically if $S' + AS = 0$, so we let $S(x) = \Phi(x)$ be the fundamental matrix solution of the homogeneous equation $L\Phi = 0$. Because $\Phi(0) = I$, we know that $S(x)$ is invertible, at lease for x near 0 (In fact, it is invertible for all x. We leave that for you).
The equation $LU = F$ is thus $SV' = F$ so $V'(x) = S^{-1}(x)F(x)$. Integrating this we can obtain the desired particular solution, U_{part} of $LU = F$. Since we just want a particular solution, we can let $U_{\text{part}}(0) = 0$, which implies $V(0) = 0$. Thus the desired formula is:

$$U_{\text{part}}(x) = S(x)V(x) = S(x)\left[V(0) + \int_0^x S^{-1}(s)F(s)\,ds\right] \tag{12}$$

$$= \int_0^x S(x)S^{-1}(s)F(s)\,ds = \int_0^x G(x,t)F(s)\,ds, \tag{13}$$

where $G(x,t) := S(x)S^{-1}(s)$ is Green's function for this problem.

EXAMPLE 1 (CONTINUED) We are now in a position to write a formula for a particular solution of $LU = F$ for Example 1 above. Then (11) is the fundamental matrix solution for the homogeneous equation, $S(x) = \Phi(x)$. Since this happens to be an orthogonal matrix, its inverse is just the transpose. Consequently

$$G(x,s) = \Phi(x)\Phi^{-1}(s) = \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} \tag{14}$$

$$= \begin{pmatrix} \cos(x-s) & \sin(x-s) \\ -\sin(x-s) & \cos(x-s) \end{pmatrix}. \tag{15}$$

Consequently

$$U_{\text{part}}(x) = \int_0^x \begin{pmatrix} \cos(x-s) & \sin(x-s) \\ -\sin(x-s) & \cos(x-s) \end{pmatrix} \begin{pmatrix} f_1(s) \\ f_2(s) \end{pmatrix} ds, \tag{16}$$

EXAMPLE 2 We can write any second order equation $u'' + bu' + cu = f$ as a first order system by letting $u_1(x) = u(x)$ and $u_2(x) = u'(x)$. Then, using the differential equation,

$$u_1' = u_2 \quad \text{and} \quad u_2' = u'' = -bu_2 - cu_1 + f.$$

that is,

$$\begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}' + \begin{pmatrix} 0 & -1 \\ c & b \end{pmatrix} \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix} = \begin{pmatrix} 0 \\ f(x) \end{pmatrix}$$

In the special case of $u'' + u = f$ we have $b = 0$ and $c = 1$ so the previous equation becomes

$$\begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}' + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix} = \begin{pmatrix} 0 \\ f(x) \end{pmatrix},$$

which is exactly (10) with $f_1 = 0$ and $f_2 = f$. Now (16) gives a formula for a particular solution of this inhomogeneous equation. It is

$$\begin{aligned} U_{\text{part}}(x) &= \int_0^x \begin{pmatrix} \cos(x-s) & \sin(x-s) \\ -\sin(x-s) & \cos(x-s) \end{pmatrix} \begin{pmatrix} 0 \\ f(s) \end{pmatrix} ds \\ &= \int_0^x \begin{pmatrix} \sin(x-s) \\ \cos(x-s) \end{pmatrix} f(s)\, ds \end{aligned}$$

Since $u_1(x) = u(x)$, this formula is exactly the same as (4) found earlier.