Math 425, Spring 2011 **Jerry L. Kazdan**

Problem Set 10

DUE: Thursday April 14 [*Late papers will be accepted until 1:00 PM Friday*].

1. This problem is to help with a computation in class today (Thursday) finding a formula for a particular solution of the inhomogeneous heat equation using an idea due to Duhamel.

   a) If $f(x,r)$ is a smooth function of the real variables t, r, let
   $$H(t,r) := \int_0^t f(x,r)\,dx.$$
   Compute $H_t(t,r)$ and $H_r(t,r)$.

   b) Let $K(t) := H(t,t)$. Use the chain rule to compute dK/dt.

2. a) Let A be a positive definite $n\times n$ real matrix, $b\in\mathbb{R}^n$, and consider the quadratic polynomial
   $$Q(x) := \tfrac{1}{2}\langle x, Ax\rangle - \langle b, x\rangle.$$
   Show that Q is bounded below, that is, there is a constant m so that $Q(x)\ge m$ for all $x\in\mathbb{R}^n$.

   b) If $x_0\in\mathbb{R}^n$ minimizes Q, show that $Ax_0=b$. [Moral: One way to solve $Ax=b$ is to minimize Q.]

   c) Let $\Omega\in\mathbb{R}^n$ be a bounded region with smooth boundary and $F(x)$ a bounded continuous function. Also, let $\mathcal{S}$ be the set of smooth functions $u(x)$ on Ω that are zero on the boundary, $u(x)=0$ for all $x\in\partial\Omega$. Define
   $$J(u) := \iint_\Omega [\tfrac{1}{2}|\nabla u|^2 + F(x)u]\,dx.$$
   If $u_0(x)\in\mathcal{S}$ minimizes $J(u)$ for all $u\in\mathcal{S}$, show that $\Delta u_0 = F$ in Ω – and of course $u_0=0$ on $\partial\Omega$. [Moral: One way to solve $\Delta u = F$ with $u=0$ on $\partial\Omega$ is to seek a function in $\mathcal{S}$ that mimimizes $J(u)$.]

3. Find a formula for the solution of $u_t = u_{xx} - u$, $x\in\mathbb{R}$ with initial conditions $u(x,0)=f(x)$ in two ways:

   a) Using Fourier Transforms.

   b) Using the procedure of Problem Set 9 #1.

4. Let $g_\lambda(\theta)$ be a continuous 2π periodic function of θ depending of the real parameter $\lambda>0$ with the properties
   $$a).\ \ g_\lambda(\theta)\ge 0, \qquad b).\ \ \int_{-\pi}^{\pi} g_\lambda(\theta)\,d\theta = 1, \qquad c).\ \ \text{For any } \delta>0,\ \ \lim_{\lambda\searrow 0}\int_{S_\delta} g_\lambda(\theta)\,d\theta = 0,$$

where S_δ is the circle $\{-\pi \le \theta \le \pi\}$ with the interval $\{|\theta| \le \delta\}$ excluded. An simple example is $g_\lambda(t) := g(t/\lambda)/\lambda$ (for $0 < \lambda \le 1$), where

$$g(t) = \begin{cases} 1 - |t| & \text{for } |t| < 1 \\ 0 & \text{for } 1 \le |t| \le \pi \end{cases}$$

and extended to $\mathbb{R}$ as a 2π periodic function.

If $f(\theta)$ is any continuous 2π periodic function, define

$$f_\lambda(\theta) := \int_{-\pi}^{\pi} f(\phi) g_\lambda(\theta - \phi)\, d\phi.$$

Show that for any θ:

$$\lim_{\lambda \searrow 0} f_\lambda(\theta) \to f(\theta).$$

Better yet, $\max_{\theta \in [-\pi,\pi]} |f_\lambda(\theta) - f(\theta)| \to 0$.

REMARK: The most important special case of this is Poisson's formula for the solution of $\Delta u = 0$ in the unit disk with boundary value $u(1,\theta) = f(\theta)$. Here with $\lambda = 1 - r$ we have

$$g_\lambda(\theta) := \frac{1 - r^2}{2\pi(1 - 2r\cos\theta + r^2)}.$$

Bonus Problem

1-B This problem constructs some smooth functions that are useful when working with partial differential equations.

a) For any integer $n \ge 0$, show that $\lim_{x \searrow 0} \dfrac{e^{-1/x}}{x^n} = 0$.

b) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{for } x > 0, \\ 0 & \text{for } x \le 0, \end{cases}.$$

Sketch the graph of f.

c) Show that f is a smooth function for all real x.

d) Show that each of the following are smooth and sketch their graphs:

$$g(x) = f(x)f(1-x) \qquad h(x) = \frac{f(x)}{f(x) + f(1-x)}$$

$$k(x) = h(x)h(4-x) \qquad K(x) = k(x+2),$$

$$\varphi(x,y) = K(x)K(y),\ (x,y) \in \mathbb{R}^2 \qquad \Phi(x) = K(\|x\|),\ x = (x_1, x_2) \in \mathbb{R}^2$$

[Last revised: April 10, 2011]