Problem Set 8

DUE: Thursday March 31 [*Late papers will be accepted until 1:00 PM Friday*].

1. a) Let A be an $n\times n$ invertible real symmetric matrix, $b\in\mathbb{R}^n$ and $c\in\mathbb{R}$. For $x\in\mathbb{R}^n$ consider the quadratic polynomial
$$Q(x)=\langle x, Ax\rangle+\langle b, x\rangle+c.$$
Show that by a translation by some vector $v\in\mathbb{R}^n$, so $x=y+v$ in the new y variable the polynomial has the form
$$Q(y)=\langle y, Ay\rangle+\gamma$$
for some real constant γ. HINT: Prove and use that for any vectors y and v we have $\langle Ay, v\rangle=\langle Av, y\rangle$.
[This generalizes "completing the square" from high school algebra.]

   b) Let $x,y\in\mathbb{R}$. Compute $\displaystyle\iint_{\mathbb{R}^2} e^{-(2x^2-2xy+3y^2+x-2y-3)}\,dx\,dy$

   c) Let $h(t)$ be a given function and say you know that $\int_0^\infty h(t)\,dt=\alpha$. If C be a positive definite real (symmetric) 2×2 matrix and $x\in\mathbb{R}^2$. Show that
$$\iint_{\mathbb{R}^2} h(\,\langle x, Cx\rangle\,)\,dA=\frac{\pi\alpha}{\sqrt{\det C}}$$
and use this to compute
$$\iint_{\mathbb{R}^2}\frac{dx\,dy}{(1+x^2+2xy+5y^2)^2},$$
where $x,y\in\mathbb{R}$.

2. Let λ_1 be the lowest eigenvalue of the $n\times n$ real symmetric matrix A. Show that
$$\lambda_1=\min_{x\neq 0}\frac{\langle x, Ax\rangle}{\|x\|^2}.$$

3. In class for $x\in\mathbb{R}^3$ we used the special function $v(x)=\dfrac{A}{|x-x_0|}$ to find a formula for a particular solution of the inhomogeneous equation $\Delta u=h(x)$:
$$u(x_0)=-\frac{1}{4\pi}\iiint_{\mathbb{R}^3}\frac{h(x)}{|x-x_0|}\,dx.$$
Use the same idea with $v(x)=A\log|x-x_0|$ to find a formula for a particular solution of the inhomogeneous equation $\Delta u=h(x)$ in the plane $\mathbb{R}^2$. Assume $u(x)$ (and hence $h(x)$) vanishes outside some sphere.

4. a) Let B be the ball $\{r^2 = x^2 + y^2 + z^2 < a^2\}$ in $\mathbb{R}^3$. Compute all the *radial* eigenfunctions $u(r)$ of $-\Delta$ with Neumann boundary conditions $\partial u/\partial r = 0$ for $r = a$. Thus, you are solving $-[u_{rr} + \frac{2}{r}u_r] = \lambda u$. [SUGGESTION: the substitution $v(r) = ru(r)$ is useful. Note it implies $v(0) = 0$.]

   b) Compute the corresponding eigenvalues (there is an explicit formula).

   c) Use this to solve the heat equation $u_t = \Delta u$ in B with $u_r = 0$ on the boundary in the special case where the initial temperature, $u(x,0) = \varphi(r)$ depends only on r. Your solution will be an infinite series. Please include a formula for finding the coefficients.

5. Find a bounded harmonic function in the *exterior* of the unit sphere $\{r > 1\}$ in $\mathbb{R}^3$ that satisfies $\partial u/\partial r = -\cos\theta$ on the boundary $r = 1$.