Math 425/525, Spring 2011 **Jerry L. Kazdan**

Periodic Solutions of ODEs

In class we discussed some aspects of periodic solutions of ordinary differential equations. From the questions I received, my presentation was not so clear. Here I'll give a detailed formal proof for the first order equation

$$u'(x) + a(x)u(x) = f(x) \tag{1}$$

where both $a(x)$ and $f(x)$ are periodic with period P, so, for instance, $a(x+P) = a(x)$ for all x.

Theorem 1 *If $u(x)$ is a solution of (1) with $u(P) = u(0)$, then u is periodic with period P, that is, $u(x+P) = u(x)$ for all x.*

Proof The key ingredient is the uniqueness assertion:
If $u(x)$ and $v(x)$ both satisfy (1) with $u(0) = v(0)$, then $u(x) = v(x)$ for all x.
We take this as a known fact.
Since u is a solution of (1) for all x. then

$$u''(x+P) + a(x+P)u(x+P) = f(x+P) \quad \text{for all} \quad x.$$

Therefore, because both a and f are periodic with period P:

$$u''(x+P) + a(x)u(x+P) = f(x) \quad \text{for all} \quad x.$$

Thus, if we let $v(x) := u(x+P)$, then

$$v'(x) + a(x)v(x) = f(x).$$

But if $u(P) = u(0)$, then $v(0) = u(0)$. Therefore by the uniqueness assertion, $v(x) = u(x)$ for all x, that is, $u(x+P) = u(x)$ for all x.

The related assertion for a solution of a second order equation is essentially identical except there we need to assume that both $u(0) = u(P)$ and $u'(0) = u'(P)$, since the corresponding uniqueness assertion for second order equations requires that.