My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this exam.

______________________________ ______________________________

Signature PRINTED NAME

Math 425
Mar. 17, 2015

Exam 1

Jerry Kazdan
12:00 – 1:20

DIRECTIONS This exam has two parts. Part A has 5 shorter problems (8 points each, so 40 points) while Part has 4 standard problems, (15 points each so total 60 points). Maximum total score is thus 100 points.
Closed book, no calculators or computers– but you may use one 3" × 5" card with notes on both sides.

Remember to silence your cellphone before the exam and keep it *out of sight* for the duration of the test period. This exam begins promptly at 12:00 and ends at 1:20; anyone who continues working after time is called may be denied the right to submit his or her exam or may be subject to other grading penalties. Please indicate what work you wish to be graded and what is scratch. *Clarity and neatness count.*

PART A: 5 problems, 8 points each (40 points total)

Score	
A-1	
A-2	
A-3	
A-4	
A-5	
B-1	
B-2	
B-3	
B-4	
Total	

A–1. The following is a table of inner products $\langle\ ,\ \rangle$ of functions f, g. and h

	f	g	h
f	4	0	8
g	0	1	3
h	8	3	50

(for example, $\langle g, h\rangle = \langle h, g\rangle = 3$).

a) Find the orthogonal projection of h into the plane E spanned by f and g. [Express your solution as linear combinations of f and g.]

b) Find an orthogonal basis for the three dimensional space spanned by f, g, and h.

A–2. Say $u_{tt} = 9u_{xx}$ for all $-\infty < x < \infty$ with $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$. Find the largest interval $J = \{a \le x \le b\}$ where modifying f and/or g *inside* this interval can change the value of $u(6,5)$.

A–3. Let $\mathcal{D} \subset \mathbb{R}^2$ be a bounded (connected) region with smooth boundary $\mathcal{B}$. If the second derivatives of $u(x,y)$ are continuous, let $Lu = u_{xx} + u_{yy} - 3u$. If in $\mathcal{D}$ s we know that u is an eigenfunction of L with corresponding eigenvalue λ, so $Lu = \lambda u$, and if $u(x,y) = 0$ on $\mathcal{B}$, show that $\lambda < 0$.

A–4. Let $f(s)$ be a twice differentiable function of the real variable s and let $u(x, y) = f(ax + by)$, where a and b are any real constants. Show that u satisfies the equation $u_{xx}u_{yy} - (u_{xy})^2 = 0$.

A–5. Describe how you would solve the heat equation $u_t = u_{xx}$ on the semi-infinite interval $x \geq 0$ with given initial temperature $u(x, 0) = f(x)$ and boundary value $u(0, t) = 0$.

Part B begins on the next page

PART B 4 traditional problems. 15 points each (so 60 points).

B–1. Say $u(x, y)$ satisfies the partial differential equation $u_x - u_y = u$.

a) Find a change of variables of the form $r = ax$, $s = x + by$, that is, find a and b, so that in these variables the equation becomes $u_r = u$.

b) Use this to find the solution of $u_x - u_y = u$ with $u(0, y) = y^2$.

B–2. Use the method of separation of variables to solve the heat equation

$$u_t = u_{xx}, \quad 0 \leq x \leq \pi, \quad t \geq 0$$

with the initial and boundary conditions

$$u(x, 0) = 2\sin 3x + \sin 7x, \quad \text{and} \quad u(0, t) = u(\pi, t) = 0.$$

B–3. For a harmonic function, so $\Delta u = u_{xx} + u_{yy} = 0$, one proof we gave of the mean value property also shows that if $u_{xx} + u_{yy} \geq 0$ in a disc centered at p with radius a, then, in polar coordinates,

$$u(p) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a, \theta)\, d\theta.$$

a) Use this to show that if $\Delta u \geq 0$ in a bounded region $\mathcal{D} \subset \mathbb{R}^2$ with boundary $\mathcal{B}$, then

$$\max_{\mathcal{D}} u(x, y) \leq \max_{\mathcal{B}} u(x, y).$$

b) Apply this to show that if $\Delta v = f$ and $\Delta w = g$ with $f \geq g$ in $\mathcal{D}$ and $v = w$ on $\mathcal{B}$, then $v \leq w$ in $\mathcal{D}$.

B–4. Say $u(x,t)$ is a solution of the modified heat equation

$$u_t = u_{xx} - 2u_x \qquad \text{for } \; 0 < x < L \; \text{ and } \; t > 0 \tag{1}$$

a) If u satisfies the initial and boundary conditions

$$u(x,0) = 0, \quad \text{and} \;\; u(0,t) = 0, \quad u(L,t) = 0 \;\; \text{for all} \;\; t \geq 0.$$

Use an "energy" argument involving

$$E(t) = \tfrac{1}{2} \int_0^L u^2(x,t)\, dx$$

to show that $u(x,t) = 0$ for all $t \geq 0$.

b) Show uniqueness of the initial/boundary value problem for solutions $w(x,t)$ of equation (1) that satisfy

$$w(x,0) = \varphi(x), \quad \text{and} \quad w(0,t) = g(t), \quad w(L,t) = h(t) \;\; \text{for all } t \geq 0.$$