Math 425 | Final Exam | Jerry L. Kazdan
May 7, 2015 | 9:00 – 11:00

DIRECTIONS This exam has two parts. Part A has 5 shorter problems (8 points each, so 40 points), while Part B has 5 traditional problems (15 points each, so total is 75 points). Maximum score is thus $40 + 75 = 115$ points.

Closed book, no calculators or computers – but you may use one sheet of $8.5'' \times 11''$ paper with notes on ONE side.

Please remember to silence your cellphone before the exam and keep it out of sight for the duration of the test period. This exam begins promptly at 9:00 and ends at 11:00; anyone who continues working after time is called may be denied the right to submit his or her exam or may be subject to other grading penalties. Please indicate what work you wish to be graded and what is scratch. *Clarity and neatness count.*

Part A 5 shorter problems, 8 points each.

A–1. Find a function $u(x,t)$ that satisfies $u_t - u = 7x$ with $u(x,0) = 0$.

A–2. Let $\Omega \subset \mathbb{R}^2$ be a bounded region with boundary $\partial\Omega$.

Say $u(x,y,t)$ is a solution of $u_t - \Delta u - 2u = e^t \sin(x+2y)$ in Ω with
$u(x,y,t) = 0$ on $\partial\Omega$ and $u(x,y,0) = 0$;

and $v(x,y,t)$ is a solution of $v_t - \Delta v - 2v = 0$ in Ω with
$v(x,y,t) = x^2 y$ on $\partial\Omega$ and $w(x,y,0) = \cos(2x)$.

Find a function $w(x,y,t)$ that satisfies $w_t = \Delta w + 2w + 3e^t \sin(x+2y)$ in Ω with
$w(x,y,t) = 5x^2 y$ on $\partial\Omega$ and $w(x,y,0) = 5\cos(2x)$.

[Your solution should give a simple formula for w in terms of u and v].

A–3. Say $u(x,t)$ is a solution of the wave equation $u_{tt} = 9u_{xx}$ for all $-\infty < x < \infty$, with $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$.

In the (x,t)-plane, find all the points on the x-axis that can influence the solution at $x = 2$, $t = 4$.

A–4. Let Ω be the unit disk in the plane $\mathbb{R}^2$. Let $-\Delta v_1 = \lambda_1 v_1$, where λ_1 is the lowest eigenvalue of the Laplacian in Ω and $v_1(x,y)$ is the corresponding eigenfunction with $v_1(x,y) = 0$ on $\partial\Omega$.

Use the inscribed and circumscribed squares for the disk to find number α and β so that $0 < \alpha < \lambda_1 < \beta$.

A–5. Let $\Omega \subset \mathbb{R}^3$ be a bounded region with smooth boundary $\partial\Omega$. Let u and v be harmonic functions in Ω with $u = f$ on $\partial\Omega$ and $v = g$ on $\partial\Omega$. If $f \geq g$, show that $u \geq v$ in Ω.

Part B 5 standard problems (15 points each, so 75 points)

B–1. Suppose $f(x) = \begin{cases} -1 & \text{for} \quad -\pi \leq x \leq 0 \\ 1 & \text{for} \quad 0 < x \leq \pi \end{cases}.$

a) Compute the Fourier series of $f(x)$ for the interval $-\pi \leq x \leq \pi$.

b) Draw a graph of the Fourier series (computed above) for $-2\pi \leq x \leq 2\pi$ and put an "X" at all points of discontinuity.

c) Give a formula relating the Fourier coefficients to $\int_{-\pi}^{\pi} f^2(x)\,dx$.

B–2. Use separation of variables to solve the wave equation $u_{tt} = u_{xx}$ for $0 \leq x \leq \pi$ with

boundary conditions: $u(0,t) = 0$ and $u_x(\pi,t) = 0$,

and

initial conditions: $u(x,0) = \sin(3x/2) - 7\sin(5x/2)$ and $u_t(x,0) = 0$.

B–3. For $-\infty < x < \infty$ and $t > 0$ consider the diffusion equation

$$u_t = u_{xx} + 2u_x + u \quad \text{with} \quad \text{u}(x,0) = e^{-x^2} \tag{1}$$

a) Show that by making the change of variables $u(x,t) = e^{ax+bt}v(x,t)$ using a clever choice of the constants a and b, the function v satisfies the standard diffusion equation

$$v_t = v_{xx}$$

but with a modified initial condition, $v(x,0)$.

b) Use this to write a formula (involving an integral) for the solution of equation (1) with the specified initial condition.

B–4. For $(x,y,z) \in \mathbb{R}^3$, let $u(x,y,z,t)$ be a solution of the Klein-Gordon equation

$$u_{tt} - \Delta u + u = 0.$$

Let

$$E(t) = \tfrac{1}{2} \iiint_{\mathbb{R}^3} \left[u_t^2 + |\nabla u|^2 + u^2\right] \, d\,\text{Vol}.$$

a) Assuming $|u(x,y,t)|$ is small for $R = \sqrt{x^2+y^2+z^2} \to \infty$, show that $E(t)$ is a constant.

b) Use this to state and prove a uniqueness result for the solution of the Klein-Gordon equation with specified initial position and velocity.

B–5. Let Ω be a bounded set in $\mathbb{R}^3$ with smooth boundary, let $f(\vec{x})$ be a smooth function on Ω and let $g(\vec{x})$ be a smooth function on $\partial\Omega$. Define

$$J(w) = \frac{1}{2} \iiint_{\Omega} \left[|\nabla w|^2 + 2fw \right] \, d\,\mathrm{Vol}.$$

If a smooth function $u(\vec{x})$ minimizes J among all smooth functions $w(\vec{x})$ for which $w = g$ on $\partial\Omega$, show that

$$\Delta u = f \quad \text{in} \quad \Omega \quad \text{and} \quad u = g \text{ on } \partial\Omega.$$