Completeness: Strauss 2^{nd} Edition, pp. 311–313

The following is a rewording of the proof in Strauss of the completeness of the eigenfunctions of the Laplacian in a region D with Dirichlet (the same proof works for Neumann boundary conditions). Although our computation is essentially identical, here we more explicitly use the elementary geometry of inner product spaces.

Say we have the eigenvalues $\lambda_1, \lambda_2, \ldots$ and corresponding orthonormal eigenfunctions $v_1, v_2, \ldots$ so $-\Delta v_j = \lambda_j v_j$, $j = 1, 2, \ldots$. Let $S_N = \operatorname{span}\{v_1, \ldots, v_N\}$ and let $P_N(f) = \sum_{k=1}^{N} c_k v_k$ be the first N terms of the "Fourier" expansion of a trial function f that is zero on the boundary of D. In geometric language, $P_N(f)$ is the orthogonal projection of f into S_N. Let

$$r_N = f - P_N(f)$$

be the remainder after N terms. We want to show that $\|r_N\| \to 0$ as $N \to \infty$. Note that $r_N \perp S_N$. Thus it is a trial function in the minimum problem for λ_{N+1}

$$\lambda_N \le \lambda_{N+1} = \min{}_{w \perp S_N} \frac{\|\nabla w\|^2}{\|w\|^2} \le \frac{\|\nabla r_N\|^2}{\|r_N\|^2}.$$

Consequently,

$$\|r_N\|^2 \le \frac{\|\nabla r_N\|^2}{\lambda_N}. \tag{1}$$

We claim that

$$\|\nabla r_N\| \le \|\nabla f\| \tag{2}$$

and that $\lambda_N \to \infty$. Using inequality (1) this will prove that $\|r_N\| \to 0$.
First observe that

$$\nabla f = \nabla[f - P_N(f)] + \nabla P_N(f) = \nabla r_N + \nabla P_N(f).$$

We claim that ∇r_N and $\nabla P_N(f)$ are orthogonal. Assuming this, then by the Pythagorean theorem

$$\|\nabla f\|^2 = \|\nabla r_N\|^2 + \|\nabla P_N(f)\|^2 \ge \|\nabla r_N\|^2,$$

which proves the inequality (2). To prove the orthogonality, by Green's first identity

$$\langle \nabla r_N, \nabla P_N(f)\rangle = \langle r_N, -\Delta P_N(f)\rangle.$$

However

$$-\Delta P_N(f) = -\Delta\Big(\sum_{k=1}^{N} c_k v_k\Big) = \sum_{k=1}^{N} c_k \lambda_k v_k \in S_N,$$

while $r_N \perp S_N$.

In Section 11.6 Strauss proves that $\lambda_N \to \infty$ by giving an explicit asymptotic formula. Using inequality (1) we conclude that $\|r_N\| \to 0$, as desired.