

The Heat and Diffusion Equations

Heat Equation

Say we have a region D with density (mass per unit volume) ρ , specific heat c and thermal conductivity k . We assume there are no internal sources of energy (such as chemical processes). Specific Heat deals with the ability of the material to regulate the (state) temperature within the materials whereas the thermal conductivity of the material deals with the ability of heat transmitted through the materials.

Specific heat is the amount of energy required to raise the body temp. by 1°C.

There is no relationship at all between thermal conductivity and specific heat. This is not only true among different elements, alloys, compounds, etc, where many examples can be shown of substances that have high thermal conductivities and low specific heat. This can also easily be demonstrated with allotropes of many common elements. For example, compare graphite and diamond. Both are pure carbon; chemically identical. The only difference is crystal structure. Graphite has a specific heat that is about 50% higher than diamond. But diamond has a thermal conductivity that is more than 800 times better than graphite.

Let $u(x, y, z, t)$ be the temperature and $H(t)$ the amount of heat (in calories) contained in a region D :

$$H(t) = \iiint_D c\rho u(x, y, z, t) dx dy dz.$$

The rate of change in this is

$$\frac{dH}{dt} = \iiint_D c\rho u_t dx dy dz.$$

Experiments show that heat flows from a hot region to a colder one at a rate proportional to the temperature gradient (*Fourier's Law*). Since heat gains or losses can only be at the boundary, ∂D of D (this is *conservation of energy*) we have

$$\frac{dH}{dt} = \iint_{\partial D} k(\mathbf{N} \cdot \nabla u) dS,$$

where k is the thermal conductivity, \mathbf{N} is the unit outer normal vector and dS is the element of surface area on the boundary. Equating the last two formulas and using the divergence theorem we find:

$$\iiint_D c\rho u_t dx dy dz = \iiint_D \nabla \cdot (k\nabla u) dx dy dz.$$

Because the region D was arbitrary we conclude that

$$c\rho u_t = \nabla \cdot (k\nabla u) \quad \text{Heat Equation.}$$

If k is a constant we write it as

$$c\rho u_t = k\Delta u,$$

where $\Delta u = u_{xx} + u_{yy} + u_{zz}$ is the *Laplacian*, which some write as $\nabla^2 u$.

Diffusion Equation

Say we have a fluid at rest in a cylindrical pipe along the x -axis and let $u(x, t)$ be the concentration (mass per unit length) of some dye in the fluid. By experiments, at a point x at time t the dye diffuses at a rate proportional to the concentration. This is *Fick's Law*.

In the section of the pipe $x_0 \leq x \leq x_1$ the Mass of the dye is

$$M(t) = \int_{x_0}^{x_1} u(x, t) dx \quad \text{so} \quad \frac{dM}{dt} = \int_{x_0}^{x_1} u_t(x, t) dx.$$

But this change in mass can only be due to the flow from the ends of the pipe so by Fick's law:

$$\frac{dM}{dt} = \text{net flow (in/out)} = ku_x(x_1, t) - ku_x(x_0, t),$$

where k is a proportionality constant depending on the fluid and the dye. Equating the above expressions for dM/dt we find:

$$\int_{x_0}^{x_1} u_t(x, t) dx = ku_x(x_1, t) - ku_x(x_0, t).$$

Now take the derivative with respect to x_1 to obtain the *diffusion equation*

$$u_t = ku_{xx}.$$

In three dimensions, similar reasoning for a region D with boundary ∂D gives

$$\iiint_D u_t dx dy dz = \iint_{\partial D} k(\mathbf{N} \cdot \nabla u) dS.$$

Just as in the above derivation of the heat equation, the divergence theorem gives the diffusion equation in three space dimensions:

$$u_t = k\Delta u.$$

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