Fritz John

Partial Differential Equations

Third Edition

Springer-Verlag New York · Heidelberg · Berlin

5. Show that for $n=1$ the solution of (1.8a, b) with $f(x)=1$ for $x>0$, $f(x)=0$ for $x<0$ is given by

$$u(x,t)=\frac{1}{2}\left[1+\phi\left(\frac{x}{\sqrt{4t}}\right)\right], \tag{1.27a}$$

where $\phi(s)$ is the "error function"

$$\phi(s)=\frac{2}{\sqrt{\pi}}\int_0^s e^{-t^2}\,dt.. \tag{1.27b}$$

6. Show that for $f(x)$ continuous and of compact support we have $\lim_{t\to\infty}u(x,t)=0$ uniformly in x for the u given by (1.11).

7. For $n=1$ let $f(x)$ be bounded, continuous, and positive for all real x.
   (a) Show that for the u given by (1.11)

$$|u(\xi+i\eta,t)|\leqslant e^{\eta^2/4t}u(\xi,t) \tag{1.28}$$

   for real ξ,η,t with $t>0$. [Hint: (1.16).]
   (b) Show that

$$|u_x(x,t)|\leqslant\frac{e^{1/2}}{\sqrt{2t}}\sup_{|y|\leqslant\sqrt{2t}}u(x+y,t) \tag{1.29}$$

   (x,y,t real, $t>0$). [Hint: Use Cauchy's expression for $u_x(x,t)$ as an integral of u over the circle of radius $\sqrt{2t}$ and center x in the complex plane.] (This gives a means to estimate the maximum possible age t of an observed heat distribution u in terms of its maximum and its gradient, assuming that it has been positive and bounded for a time t.)

8. Find all solutions $u(x,t)$ of the one-dimensional heat equation $u_t=u_{xx}$ of the form

$$u=\frac{1}{\sqrt{t}}f\left(\frac{x}{2\sqrt{t}}\right).$$

[Hint: $f(z)$ has to satisfy a linear ordinary second-order equation, of which one solution $f(z)=e^{-z^2}$ is known, from $u=K(x,0,t)$. All others can then be found by quadratures.]

(b) *Maximum principle, uniqueness, and regularity*

Let ω denote an open bounded set of $\mathbb{R}^n$. For a fixed $T>0$ we form the cylinder Ω in $\mathbb{R}^{n+1}$ with base ω and height T:

$$\Omega=\{(x,t)|x\in\omega,\ 0<t<T\}. \tag{1.30a}$$

The boundary $\partial\Omega$ consists of two disjoint portions, a "lower" boundary $\partial'\Omega$, and an "upper" one $\partial''\Omega$ (see Figure 7.1):

$$\partial'\Omega=\{(x,t)|\text{either } x\in\partial\omega,\ 0\leqslant t\leqslant T \text{ or } x\in\omega,\ t=0\} \tag{1.30b}$$

$$\partial''\Omega=\{(x,t)|x\in\omega,\ t=T\}. \tag{1.30c}$$

As in the second-order elliptic case the maximum of a solution of the heat equation in Ω is taken on $\partial\Omega$; but a more subtle distinction between the forward and backwards t-directions makes itself felt:

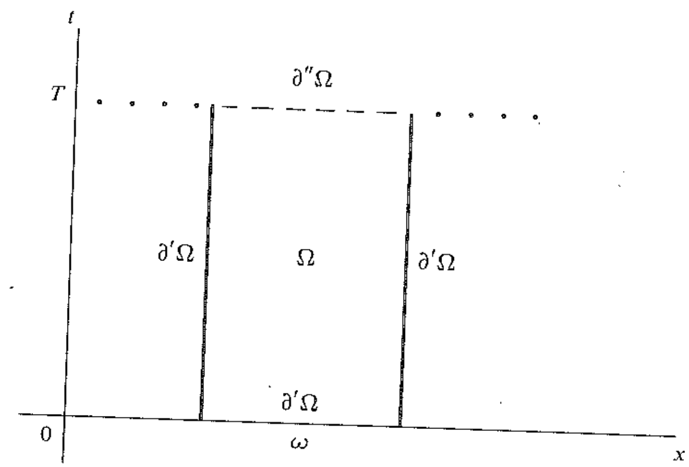

Figure 7.1

Theorem. *Let u be continuous in $\bar{\Omega}$ and $u_t, u_{x_ix_k}$ exist and be continuous in Ω and satisfy $u_t-\Delta u\leqslant 0$. Then*

$$\max_{\bar{\Omega}} u=\max_{\partial'\Omega} u. \tag{1.31}$$

PROOF. Let at first $u_t-\Delta u<0$ in Ω. Let Ω_ε for $0<\varepsilon<T$ denote the set

$$\Omega_\varepsilon=\{(x,t)|x\in\omega,\ 0<t<T-\varepsilon\}.$$

Since $u\in C^0(\bar{\Omega}_\varepsilon)$ there exists a point $(x,t)\in\bar{\Omega}_\varepsilon$ with

$$u(x,t)=\max_{\bar{\Omega}_\varepsilon} u.$$

If here $(x,t)\in\Omega_\varepsilon$ the necessary relations $u_t=0$, $\Delta u\leqslant 0$ would contradict $u_t-\Delta u<0$. If $(x,t)\in\partial''\Omega_\varepsilon$ we would have

$$u_t\geqslant 0,\qquad \Delta u\leqslant 0$$

leading to the same contradiction. Thus $(x,t)\in\partial'\Omega_\varepsilon$, and

$$\max_{\bar{\Omega}_\varepsilon} u=\max_{\partial'\Omega_\varepsilon} u\leqslant\max_{\partial'\Omega} u.$$

Since every point of $\bar{\Omega}$ with $t<T$ belongs to some $\bar{\Omega}_\varepsilon$ and u is continuous in $\bar{\Omega}$, (1.31) follows. Let next $u_t-\Delta u\leqslant 0$ in Ω. Introduce

$$v(x,t)=u(x,t)-kt$$

with a constant positive k. Then $v_t-\Delta v=u_t-\Delta u-k<0$ and

$$\max_{\bar{\Omega}} u=\max_{\bar{\Omega}}(v+kt)\leqslant\max_{\bar{\Omega}} v+kT=\max_{\partial'\Omega} v+kT\leqslant\max_{\partial'\Omega} u+kT.$$

For $k\to 0$ we obtain (1.31). □

The maximum principle immediately yields a uniqueness theorem

Theorem. *Let u be continuous in $\bar{\Omega}$ and $u_t, u_{x_i x_k}$ exist and be continuous in Ω. Then u is determined uniquely in $\bar{\Omega}$ by the value of $u_t - \Delta u$ in Ω and of u on $\partial'\Omega$.*

For the proof it is sufficient to consider the case where $u_t - \Delta u = 0$ in Ω and $u = 0$ on $\partial'\Omega$. Applying (1.31) to u and $-u$ we find that

$$\max_{\bar{\Omega}} u = \max_{\bar{\Omega}} (-u) = 0, \tag{1.32}$$

and hence that $u=0$ in $\bar{\Omega}$.

We can extend the maximum principle and the uniqueness theorem to the case where Ω is the "slab"

$$\Omega = \{(x,t) \mid x \in \mathbb{R}^n, 0 < t < T\}, \tag{1.33}$$

if we assume that u satisfies a certain growth condition at infinity.

Theorem. *Let u be continuous for $x \in \mathbb{R}^n$, $0 \leqslant t \leqslant T$, and let $u_t, u_{x_i x_k}$ exist and be continuous for $x \in \mathbb{R}^n$, $0 < t < T$, and satisfy*

$$u_t - \Delta u \leqslant 0 \quad \text{for } 0 < t < T, x \in \mathbb{R}^n \tag{1.34a}$$

$$u(x,t) \leqslant Me^{a|x|^2} \quad \text{for } 0 < t < T, x \in \mathbb{R}^n \tag{1.34b}$$

$$u(x,0) = f(x) \quad \text{for } x \in \mathbb{R}^n.$$

Then

$$u(x,t) \leqslant \sup_z f(z) \quad \text{for } 0 \leqslant t \leqslant T, x \in \mathbb{R}^n. \tag{1.35}$$

It is clear that this theorem implies that the solution of the initial-value problem

$$u_t - \Delta u = 0 \quad \text{for } 0 < t < T \tag{1.36a}$$

$$u(x,0) = f(x) \tag{1.36b}$$

is unique provided we restrict ourselves to solutions satisfying

$$|u(x,t)| \leqslant Me^{a|x|^2} \quad \text{for } 0 < t < T. \tag{1.36c}$$

This shows that for bounded continuous f formula (1.11) represents the only *bounded* solution u of (1.8a, b). Obviously the Tychonoff solution (1.19), (1.20) for which $u(x,0)=0$ cannot satisfy an inequality of the type (1.36c). By (1.24) it does satisfy such an inequality with the constant a replaced by $1/\theta t$.

Proof of the theorem. It is sufficient to show (1.35) under the assumption that

$$4aT < 1 \tag{1.37a}$$

For we can always divide the interval $0 \leqslant t \leqslant T$ into equal parts, each of length $\tau < 1/4a$, and conclude successively for $k = 0, 1, \ldots, T/\tau$ that

$$u(x,t) \leqslant \sup_y u(y, k\tau) \leqslant \sup_y u(y,0)$$

for $k\tau \leqslant t \leqslant (k+1)\tau$. Assume then (1.37a). We can find an $\varepsilon > 0$ such that

$$4a(T+\varepsilon) < 1. \tag{1.37b}$$

Given a fixed y we consider for constants $\mu > 0$ the functions

$$\begin{aligned} v_\mu(x,t) &= u(x,t) - \mu(4\pi(T+\varepsilon-t))^{-n/2} \exp\left[|x-y|^2/4(T+\varepsilon-t)\right] \\ &= u(x,t) - \mu K(ix, iy, T+\varepsilon-t) \end{aligned} \tag{1.38}$$

defined for $0 \leqslant t \leqslant T$. Since $K(x,y,t)$ as defined by (1.10d), with $|x-y|^2$ replaced by $(x-y)\cdot(x-y)$, satisfies $K_t = \Delta K$ for any complex x, y, t with $t \neq 0$, we find that

$$\frac{\partial}{\partial t} v_\mu - \Delta v_\mu = u_t - \Delta u \leqslant 0. \tag{1.39}$$

Consider the "circular" cylinder

$$\Omega = \{(x,t) \mid |x-y| < \rho, 0 < t < T\} \tag{1.40}$$

of radius ρ. Then by (1.31)

$$v_\mu(y,t) \leqslant \max_{\partial'\Omega} v_\mu. \tag{1.41}$$

Here on the plane part of $\partial'\Omega$, since $\mu K > 0$,

$$v_\mu(x,0) \leqslant u(x,0) \leqslant \sup_z f(z). \tag{1.42a}$$

On the curved part $|x-y| = \rho$, $0 \leqslant t \leqslant T$ of $\partial'\Omega$ by (1.38), (1.34b), (1.37b)

$$\begin{aligned} v_\mu(x,t) &= Me^{a|x|^2} - \mu(4\pi(T+\varepsilon-t))^{-n/2} \exp\left[\rho^2/4(T+\varepsilon-t)\right] \\ &\leqslant Me^{a(|y|+\rho)^2} - \mu(4\pi(T+\varepsilon))^{-n/2} e^{\rho^2/4(T+\varepsilon)} \\ &\leqslant \sup_z f(z) \end{aligned}$$

for all sufficiently large ρ. Thus

$$\max_{\partial'\Omega} v_\mu \leqslant \sup f(z).$$

It follows from (1.41), (1.38) that

$$v_\mu(y,t) = u(y,t) - \mu(4\pi(T+\varepsilon-t))^{-n/2} \leqslant \sup f(z)$$

For $\mu \to 0$ we obtain (1.35). □

In order to derive regularity properties of a solution of the heat equation in a bounded region we make use of Green's identity, as was done for harmonic functions on p. 76. Let again Ω denote the cylindrical region (1.30a), where ω is a bounded open set in $\mathbb{R}^n$ with sufficiently regular boundary. Let $u, u_t, u_{x_i x_k}$ exist and be continuous in $\bar{\Omega}$ and satisfy $u_t - \Delta u = 0$. For an arbitrary function $v(x,t) \in C^2(\bar{\Omega})$ we find by integration by parts that

$$\begin{aligned} 0 &= \int_\Omega v(u_t - \Delta u)\,dx \\ &= -\int_\Omega u(v_t + \Delta v)\,dx + \int_{\substack{x\in\omega \\ t=T}} vu\,dx - \int_{\substack{x\in\omega \\ t=0}} vu\,dx \\ &\quad - \int_0^T dt \int_{x\in\partial\omega} \left(v\frac{du}{dn} - u\frac{dv}{dn} \right) dS_x. \end{aligned} \tag{1.43}$$

For a certain $\xi \in \omega$ and $\varepsilon > 0$ we choose

$$v(x,t) = K(x,\xi,T+\varepsilon-t), \tag{1.44}$$

so that $v_t + \Delta v = 0$. Then for $\varepsilon \to 0$

$$\int_{\substack{x\in\omega \\ t=T}} vu\,dx = \int_{x\in\omega} K(x,\xi,\varepsilon)u(x,T)\,dx \to u(\xi,T), \tag{1.44a}$$

since by the theorem of p. 169

$$w(\xi,\varepsilon) = \int K(\xi,x,\varepsilon)u(x,T)\,dx = \int K(x,\xi,\varepsilon)u(x,T)\,dx$$

is a solution of $w_\varepsilon - \Delta_\xi w = 0$ with initial values*

$$w(\xi,0) = u(x,T).$$

Since also $K(x,\xi,T+\varepsilon-t)$ is uniformly continuous in ε, x, t for $\varepsilon \geqslant 0$, $x \in \partial\omega$, $0 \leqslant t \leqslant T$ and for $x \in \omega$, $t = 0$, we find from (1.43) that

$$\begin{aligned} u(x,T) &= \int_\omega K(x,\xi,T)u(x,0)\,dx \\ &+ \int_0^T dt \int_{x\in\partial\omega} \left(K(x,\xi,T-t)\frac{du(x,t)}{dn} - u(x,t)\frac{dK(x,\xi,T-t)}{dn} \right) dS_x. \end{aligned} \tag{1.45}$$

We use this formula to extend $u(\xi,T)$ to complex ξ-arguments $\xi = \eta + i\zeta$ (with η, ζ real), keeping T real. The first integral in (1.45) trivially is an entire analytic function of ξ. Moreover for $0 \leqslant t < T$, $x \neq \eta$

$$K(x,\xi,T-t) = (4\pi(T-t))^{-n/2} \exp\left[-(x-\xi)\cdot(x-\xi)/4(T-t) \right]$$

is analytic in ξ and (see (1.16)) bounded in absolute value by

$$(4\pi(T-t))^{-n/2} \exp\left[\frac{|\zeta|^2 - |x-\eta|^2}{4(T-t)} \right].$$

*The fact that x is integrated only over the region ω instead over all of $\mathbb{R}^n$ does not change the proof given on p. 169, as long as ξ is a fixed point of ω.

Thus $K(x,\xi,T-t)$ is bounded uniformly for complex $\xi = \eta + i\zeta$ as long as $|x-\eta|^2 - |\zeta|^2$ is bounded below by a positive constant. The same holds for dK/dn. In the second integral we first extend the t-integration from 0 to $T-\varepsilon$ and then let $\varepsilon \to 0$. Since sequences of analytic functions which converge uniformly in a complex region have analytic limits, it follows that $u(\xi,t)$ is analytic in ξ, as long as $|x-\eta|^2 - |\zeta|^2 > 0$ for all $x \in \partial\omega$. This is certainly the case for complex ξ near a real point of ω.

It follows that $u(\xi,T)$ is real analytic for $\xi \in \omega$. More precisely $u(\xi,T)$ is analytic for those complex ξ for which $|\mathrm{Im}\,\xi|$ is less than the distance of $\mathrm{Re}\,\xi$ from $\partial\omega$. Hence a solution $u(x,t)$ of $u_t - \Delta u = 0$ is real analytic in x in any open set where u_t and the $u_{x_i x_k}$ are continuous. Moreover $u(x,t)$ will be an entire function of x if defined for all real x and for t restricted to an open interval.

We easily conclude that solutions of $u_t - \Delta u = 0$ in an open set $\Omega \in \mathbb{R}^{n+1}$ belong to $C^\infty(\Omega)$. For if u has continuous x-derivatives of all orders, then $u_{tx_k} = \Delta u_{x_k}$ is continuous, and hence equals $u_{x_k t}$. Thus $v = u_{x_k}$ also is a solution of the heat equation with the same regularity properties as u. The same holds again for $v_{x_k} = u_{x_k x_k}$ and then also for $\Delta u = u_t$. Thus $u_{tt} = \Delta^2 u$ is continuous. Proceeding in this manner yields that all derivatives of $u(x,t)$ are continuous. As observed earlier analyticity of $u(x,t)$ with respect to t cannot be expected. This fits in with the idea that the future of a heat distribution does not depend exclusively on the past, but also on outside influences that cannot be predicted.

Problem

Let u be a solution of the one-dimensional heat equation $u_t = u_{xx}$ in an open subset Ω of the xt-plane. Show that at a point of Ω there exist constants A, M such that

$$\left| \frac{\partial^k u}{\partial t^k} \right| \leqslant AM^k (2k)! \tag{1.46}$$

for all nonnegative integers k. [Hint: Use that u is analytic in x.]

(c) *A mixed problem*

For $n = 1$ let $u(x,t)$ be a solution of $u_t - u_{xx} = 0$ in a half strip

$$0 < x < L \qquad 0 < t. \tag{1.47}$$

We seek the u satisfying the boundary conditions

$$u(0,t) = u(L,t) = 0 \quad \text{for } t > 0 \tag{1.48a}$$

and initial condition

$$u(x,0) = f(x) \quad \text{for } 0 < x < L. \tag{1.48b}$$

Here u might represent the temperature in an insulated rod with the ends held at a constant temperature. This problem could be solved by Fourier