

Problem Set 0

DUE: NEVER

This is rust remover. Essentially *all* of the ideas needed for these problems should have been covered in the prerequisite courses. No messy computations are needed – but some serious thinking may be needed. NOTATION: $u_t = \frac{\partial u}{\partial t}$.

1. Let $u(t)$ be the solution of $u' = 3u$ with initial value $u(0) = A > 0$. At what time T is $u(T) = 2A$?
2. Let $u(t)$ be the amount of a radioactive element at time t and say initially, $u(0) = A > 0$. The rate of decay is proportional to the amount present, so

$$\frac{du}{dt} = -cu,$$

where the constant $c > 0$ determines the decay rate. The *half-life* T is the amount of time for half of the element to decay, so $u(T) = \frac{1}{2}u(0)$. Find c in terms of T and obtain a formula for $u(t)$ in terms of T .

3. Let $\int_0^x f(t) dt = e^{\cos(3x)} + A$, where f is some continuous function. Find f and the constant A .
4. a) If $u'' + 4u = 0$ with initial conditions $u(0) = 1$ and $u'(0) = -2$, compute $u(t)$.
 b) Find a particular solution of the inhomogeneous equation $u'' + 4u = 8$.
 c) Find a particular solution of the inhomogeneous equation $u'' + 4u = -4t$.
 d) Find a particular solution of the inhomogeneous equation $u'' + 4u = -8 - 8t$.
 e) Find the most general solution of the inhomogeneous equation $u'' + 4u = 8 - 8t$.
 f) If $f(t)$ is any continuous function, use the method “variation of parameters” (look it up if you don’t know it) to find a formula for a particular solution of $u'' + 4u = f(t)$.
5. Let $u(t)$ be any solution of $u'' + 2bu' + 4u = 0$. If $b > 0$ is a constant, show that $\lim_{t \rightarrow \infty} u(t) = 0$.
6. a) If $u'' - 4u = 0$ with initial conditions $u(0) = 1$ and $u'(0) = -2$, compute $u(t)$.
 b) Find a particular solution of the inhomogeneous equation $u'' - 4u = 8$.
 c) Find a particular solution of the inhomogeneous equation $u'' - 4u = -4t$.

- d) Find a particular solution of the inhomogeneous equation $u'' - 4u = -8 - 8t$.
- e) Find the most general solution of the inhomogeneous equation $u'' - 4u = 8 - 8t$.
- f) If $f(t)$ is any continuous function, use the method “variation of parameters” (look it up if you don’t know it) to find a formula for a particular solution of $u'' - 4u = f(t)$.

7. Say $w(t)$ satisfies the differential equation

$$aw''(t) + bw' + cw(t) = 0, \quad (1)$$

where a and c , are positive constants and $b \geq 0$. Let $E(t) = \frac{1}{2}[aw'^2 + cw^2]$.

- a) Without solving the differential equation, show that $E'(t) \leq 0$.
 - b) Use this to show that If you also know that $w(0) = 0$ and $w'(0) = 0$, then $w(t) = 0$ for all $t \geq 0$.
 - c) [Uniqueness] Say the functions $u(t)$ and $v(t)$ both satisfy the same equation (1) and also $u(0) = v(0)$ and $u'(0) = v'(0)$. Show that $u(t) = v(t)$ for all $t \geq 0$.
8. Say $u(x, t)$ has the property that $\frac{\partial u}{\partial t} = 2$ for all points $(x, t) \in \mathbb{R}^2$.
- a) Find some such function $u(x, t)$.
 - b) Find the most general such function $u(x, t)$.
 - c) If $u(x, 0) = \sin 3x$, find $u(x, t)$.
 - d) If instead u satisfies $\frac{\partial u}{\partial t} = 2xt$, still with $u(x, 0) = \sin 3x$, find $u(x, t)$.

9. Say $u(x, t)$ has the property that $\frac{\partial u}{\partial t} = 3u$ for all points $(x, t) \in \mathbb{R}^2$.

- a) Find some such function – other than the trivial $u(x, t) \equiv 0$.
 - b) Find the most general such function.
 - c) If $u(x, t)$ also satisfies the initial condition $u(x, 0) = \sin 3x$, find $u(x, t)$.
10. a) If $u(x, t) = \cos(x - 3t) + 2(x - 3t)^7$, show that $3u_x + u_t = 0$.
- b) If $f(s)$ is any smooth function of s and $u(x, t) = f(x - 3t)$, show that $3u_x + u_t = 0$.
11. A function $u(x, y)$ satisfies $3u_x + u_t = 0$. Find an invertible linear change of variables

$$\begin{aligned} r &= ax + bt \\ s &= cx + dt, \end{aligned}$$

where a, b, c, d are constants, so that in the new (r, s) variables u satisfies $\frac{\partial u}{\partial s} = 0$.

[REMARK: There are many possible such changes of variable. The point is to reduce $3u_x + u_t = 0$ to the much simpler $u_s = 0$.]

12. Let S and T be linear spaces, such as \mathbb{R}^3 and \mathbb{R}^7 and $L : S \rightarrow T$ be a *linear map*; thus, for any vectors X, Y in S and any scalar c

$$L(X + Y) = LX + LY \quad \text{and} \quad L(cX) = cL(x).$$

Say V_1 and V_2 are (distinct!) solutions of the equation $LX = Y_1$ while W is a solution of $LX = Y_2$. Answer the following in terms of V_1, V_2 , and W .

- Find some solution of $LX = 2Y_1 - 7Y_2$.
- Find another solution (other than W) of $LX = Y_2$.

13. The following is a table of inner (“dot”) products of vectors \mathbf{u} , \mathbf{v} , and \mathbf{w} .

	\mathbf{u}	\mathbf{v}	\mathbf{w}
\mathbf{u}	4	0	8
\mathbf{v}	0	1	3
\mathbf{w}	8	3	50

For example, $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v} = 3$.

- Find a unit vector in the same direction as \mathbf{u} .
 - Compute $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w})$.
 - Compute $\|\mathbf{v} + \mathbf{w}\|$.
 - Find the orthogonal projection of \mathbf{w} into the plane E spanned by \mathbf{u} and \mathbf{v} . [Express your solution as linear combinations of \mathbf{u} and \mathbf{v} .]
 - Find a unit vector orthogonal to the plane E .
 - Find an orthonormal basis of the three dimensional space spanned by \mathbf{u} , \mathbf{v} , and \mathbf{w} .
14. Let z and w be complex numbers.
- Write the complex number $z = \frac{1}{3 + 4i}$ in the form $z = a + ib$ where a and b are real numbers.
 - Show that $\overline{(zw)} = \bar{z}\bar{w}$.
 - Show that $|z|^2 = z\bar{z}$.
 - show that $|zw| = |z||w|$.

15. If $z = x + iy$ is a complex number, one way to define e^z is by the power series

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^k}{k!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}. \quad (2)$$

- a) Using the usual (real) power series for $\cos y$ and $\sin y$, show that

$$e^{iy} = \cos y + i \sin y.$$

- b) Use this to show that $\cos y = \frac{e^{iy} + e^{-iy}}{2}$ and $\sin y = \frac{e^{iy} - e^{-iy}}{2i}$.
- c) Using equation (2), one can show that $e^{z+w} = e^z e^w$ for any complex numbers z and w (accept this for now). Consequently

$$e^{i(x+y)} = e^{ix} e^{iy}.$$

Use the result of part (a) to show that this implies the usual formulas for $\cos(x+y)$ and $\sin(x+y)$.

16. Let $\mathcal{D} \subset \mathbb{R}^2$ be a bounded (connected) region with smooth boundary \mathcal{B} . If $u(x, y)$ is a “smooth” function, write $\Delta u = u_{xx} + u_{yy}$ (we call Δ the *Laplace operator*).

SUGGESTION: First do this problem for a function of *one* variable, $u(x)$, so $\Delta u = u''$ and, say, \mathcal{D} is the interval $\{0 < x < 1\}$.

- a) Show that $u\Delta u = \nabla \cdot (u\nabla u) - |\nabla u|^2$.
- b) If $u(x, y) = 0$ on \mathcal{B} . Show that

$$\iint_{\mathcal{D}} u\Delta u \, dx \, dy = - \iint_{\mathcal{D}} |\nabla u|^2 \, dx \, dy.$$

- c) If $\Delta u = 0$ in \mathcal{D} and $u = 0$ on the boundary \mathcal{B} , show that $u(x, y) = 0$ throughout \mathcal{D} .

[Last revised: January 23, 2015]