Math 425, Spring 2015 **Jerry L. Kazdan**

Problem Set 11

Due: Thurs. Apr. 16 in class. [Late papers will be accepted until 1:00 PM Friday.]

This week: Please read Chapter 11 in the Strauss text.

1. Strauss p. 247 #3

2. Strauss p. 248 #9

3. [See Section 10.1 in Strauss] Solve the wave equation $u_{tt} = c^2 \Delta u$ in the square $\Omega = \{0 < x < \pi,\ 0 < y < \pi\}$ in the plane with $\nabla u \cdot N = 0$ on the boundary and initial conditions $u(x, y, 0) = 0,\ \ u_t(x, y, 0) = 3\cos x + \cos 2x \cos 5y$.

4. Strauss p. 264 #2

   Suggestion: As a warm-up, use the simpler initial conditions:

$$u(x, y, 0) = \sin\frac{2\pi x}{a}\sin\frac{5\pi y}{b}, \quad u_t(x, y, 0) = 0.$$

   Then find the Fourier expansion in $0 \le x \le a$ of $x(a - x) = \sum c_k \sin(k\pi x/a)$.]

5. Strauss p. 264 #3

6. Strauss p. 270 #1 [book typo: should be "initial conditions (25)"]

7. Strauss p. 270 #2

8. Strauss p. 270 #5

9. On the interval $\alpha \le x \le \beta$ let $u(x)$ and $v(x)$ be solutions of the equations

$$u'' + a(x)u = 0 \qquad v'' + b(x)v = 0,$$

   where $a(x)$ and $b(x)$ are continuous functions.

   a) Show that

$$\int_\alpha^\beta (b - a)uv\, dx = \left[vu' - uv'\right]_\alpha^\beta.$$

b) Suppose that α and β are consecutive zeroes of u and that $u(x) > 0$ in the interval $\alpha < x < \beta$. If $a(x) \leq b(x)$ in this interval, show that$v(x)$ must be zero somewhere in this interval by using that $u'(\alpha) > 0$ and $u'(\beta) < 0$ and there is a contradiction unless v is zero somewhere in this interval. This is the *Sturm oscillation theorem.*

c) In the special case where $a(x) = b(x)$ and $u(x)$ and $v(x)$ are linearly independent solutions of the same equation, conclude that between any two zeroes of u there is a zero of v, and vice versa. Thus the zeroes interlace. A special case of this is the interlacing of the zeroes of $\sin x$ and $\cos x$.

d) If $b(x) \geq c^2 > 0$ for some constant c, show that $v(x)$ must have infinitely many zeroes by comparing v with a collation of $u'' + c^2 u = 0$.

e) Show that every solution of $v'' + (1 - \frac{1}{x^2} v = 0$ must have infinitely many zeroes .

[Last revised: April 20, 2015]