2) $$u(x,t) = \frac{1}{4\pi c^2}\iiint \frac{f\left(\xi, t - \frac{|\xi - x|}{c}\right)}{|\xi - x|}\, d\xi$$

let $\xi' = \xi - x$

$$= \frac{1}{4\pi c^2}\iiint_{\{|\xi'| \le ct\}} \frac{f\left(|\xi' + x|, t - \frac{|\xi'|}{c}\right)}{|\xi'|}\, d\xi'$$

$$= \frac{1}{4\pi c^2}\int_0^{ct}\int_0^{2\pi}\int_0^{\pi} \frac{f(|\xi + x|, t - r/c)}{r}\, r^2 \sin\varphi \, d\varphi \, d\theta \, dr$$

as f radial, so is u. So WLOG, let $x = (0, 0, r_0)$

Then, $|\xi + x| = \sqrt{r_0^2 + r^2 + 2 r r_0 \cos\varphi}$

$$u(x,t) = \frac{1}{4\pi c^2}\int_0^{ct}\int_0^{2\pi}\int_{-1}^{1} \frac{f\left(\sqrt{r_0^2 + r^2 + 2 r r_0 s}, t - r/c\right)}{r}\, r^2 \, ds \, d\theta \, dr$$

where we let $s = \cos\varphi$
$ds = -\sin\varphi \, d\varphi$.

$$= \frac{1}{2c^2}\int_0^{ct}\int_{-1}^{1} r f\left(\sqrt{r_0^2 + r^2 + 2 r r_0 s}, t - r/c\right) ds \, dr$$

let $p^2 = r_0^2 + r^2 + 2 r r_0 s \qquad \tau = t - r/c$

$$\left|\frac{\partial(p,\tau)}{\partial(s,r)}\right| = \begin{vmatrix} 2 r r_0 & 0 \\ 2r + 2r_0 s & -1/c \end{vmatrix} = \frac{2 r r_0}{c}$$

$$\Rightarrow u(x,t) = \frac{1}{2 c r_0}\int_0^{t}\int_{r_0 - ct + c\tau}^{r_0 + ct - c\tau} f(p,\tau)\, p \, dp \, d\tau$$

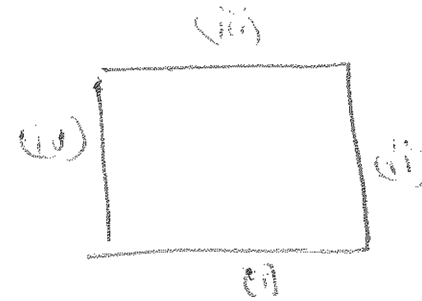
3)

$u(x,y,t) = T(t)V(x,y)$

$$\frac{T''}{c^2 T} = \frac{\Delta V}{V} = -\lambda$$

$$\boxed{T_n(t) = A_n \cos(\sqrt{\lambda_n}\, ct) + B_n \sin(\sqrt{\lambda_n}\, ct)}$$

For $V(x,y) = X(x)Y(y)$



$\Delta V + \lambda V = 0$

$\Rightarrow X''Y + XY'' = -\lambda XY$

$\Rightarrow \dfrac{X''}{X} + \dfrac{Y''}{Y} = -\lambda$

$\nabla V \cdot N = 0 \quad , \nabla V = \langle X'Y, XY' \rangle$

On (i) $y = 0$, $\hat{N} = \langle 0, -1 \rangle$

$\Rightarrow -XY'(0) = 0 \quad \Rightarrow Y'(0) = 0$

$Y_n(y) = C_n \sin(\beta_n^{(y)} y) + D_n \cos(\beta_n^{(y)} y)$

$Y_n'(y) = \beta_n^{(y)} C_n \cos(\quad) - \beta_n^{(y)} D_n \sin(\quad)$

$\therefore$ as $Y'(0) = 0$, $C_n = 0$

On (iii) $y = \pi$ $N = \langle 0, 1 \rangle$

$XY'(\pi) = 0 \quad Y'(\pi) = 0$

$\beta_n^{(y)} D_n \sin(\beta_n^{(y)} y) = 0 \quad \Rightarrow \beta_n^{(y)} = n.$

(iii) + (iv) similar

$$X_m(x) = C_m \cos(mx)$$
$$Y_k(x) = D_k \cos(ky)$$

$$\lambda_{mk} = m^2 + k^2$$

$$\therefore u(x,y,t) = \sum_{m,k} A_{mk}\left(B_\lambda \cos(\sqrt{\lambda_{mk}}\, ct) + C_\lambda \sin\left(\sqrt{\lambda_{mk}}\, ct\right)\right)$$
$$\cos(mx)\cos(ky)$$

As $u(x,y,0) = 0 \Rightarrow B_\lambda = 0$

$$\therefore u(x,y,t) \sum A_{mk} \sin(\sqrt{\lambda_{mk}}\, ct)\cos(mx)\cos(ky)$$

$$u_t(x,y,t) = \sum A_{mk}\, c\sqrt{m^2+k^2}\cos(\quad)\cos(mx)\cos(ky)$$

at $t=0$ $= \sum A_{mk}\, c\sqrt{m^2+k^2}\cos(mx)\cos(ky)$

$= 3\cos x + \cos(2x)\cos(5y)$

$A_{mk} = 0$ except

$m=1$ $k=0$

$$A_{1,0} = \frac{3}{c}$$

$m=2$ $k=5$

$$A_{2,5} = \frac{1}{\sqrt{29}\, c}$$

4) $\frac{T''}{c^2 T} = \frac{\Delta V}{V} = -\lambda$

T is sin and cos.

with $T'(0)=0$ (initial condition)

$T(t) = A_1 \cos(\sqrt{\lambda} c t)$

$\frac{X''}{X} + \frac{Y''}{Y} = -\lambda$

$u = 0$ on ∂R

$X(a) = X(0) = 0$

$Y(b) = Y(0) = 0$

$X(x) = C\cos(kx) + D\sin(kx)$

$X(0) = 0 \Rightarrow C = 0$

$X(a) = D\sin(ka) \Rightarrow ka = n\pi$

$k = \frac{n\pi}{a}$

Similar for Y.

$X(x) = C_n \sin\left(\frac{n\pi x}{a}\right) \quad Y_m(y) = D_m \sin\left(\frac{m\pi y}{b}\right)$

$\lambda_{nm} = \pi^2\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right)$

$$\Rightarrow u(x,y,t) = \sum_{n,m} A_{nm} \cos\left(\sqrt{\lambda_{nm}}\, ct\right) \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right)$$

$$u(x,y,0) = \sum_{n,m} A_{nm} \sin(\ )\sin(\ ) = xy(b-y)(a-x).$$

$$\Rightarrow A_{np} = \frac{\int_0^a \int_0^b xy(b-y)(a-x)\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)\,dy\,dx}{\int_0^a\int_0^b \sin^2\left(\frac{n\pi}{a}x\right)\sin^2\left(\frac{m\pi}{b}y\right)\,dy\,dx}.$$

5) Want ~~t~~

$u(x,t) \to 0$ as $t \to \infty$

only care about T part.

$$u = V(x,y,z)T(t) \quad \Rightarrow T'\cancel{V} = K\Delta V\, T + \gamma T V$$

$$\frac{T'}{T} - \gamma = \frac{K\Delta V}{V} = -\lambda$$

$$\Rightarrow \frac{T'}{kT} - \frac{\gamma}{k} = -\lambda$$

$$\Rightarrow T(t) = e^{(\gamma-\lambda)kt}.$$

so want $\gamma < \lambda.\ \forall \lambda.$

As in (4), $X_m(x) = \sin\left(\frac{m\pi}{a}x\right)$ $\quad Y_n(y) = \sin\left(\frac{n\pi}{a}y\right)$ $\quad Z_k(z) = \sin\left(\frac{k\pi}{a}z\right)$

$\therefore \frac{\pi^2}{a^2}(k^2+m^2+n^2) = \lambda_{mnk}$ for $k,m,n \in \mathbb{Z}^+$

$\therefore$ smallest λ is $\lambda_{111} = \frac{3\pi^2}{a^2}$ $\quad \therefore \gamma < \frac{3\pi^2}{a^2}$

6) As radial all Θ terms 0.

so $A_{nm}, B_{nm}, C_{nm}, D_{nm} = 0$ for $n \neq 0$.

For $n = 0$

as $u(x, y, 0) = 0$

$A_{0m} = 0$

7) u radial as in (6).

$$u_t^{(0)} = \sum_m J_0(\sqrt{\lambda_{0m}}\, r)\left(C_{0m}\sqrt{\lambda_{0m}}\, c\, t(0)\right)$$

$$\Rightarrow C_{0m} = C_{nm} = D_{nm} = 0.$$

$$\Rightarrow u(x, y, t) = \sum_{m=1}^{\infty} J_0\left(\sqrt{\lambda_{0m}}\, r\right) A_{0m} \cos\left(\sqrt{\lambda_{0m}}\, c t\right)$$

$$A_{0m} = \frac{1}{2\pi j_{0m}} \int_0^a \int_{-\pi}^{\pi} \left(1 - \frac{r^2}{a^2}\right) J_0(\beta_{0m} r)\, r \, d\theta \, dr$$

9) $u'' + a(x)u = 0$

$v'' + b(x)v = 0$

a) $$\int_\alpha^\beta (b-a)uv\,dx = \int_\alpha^\beta (b-a)uv\,dx$$

$$= \int_\alpha^\beta bvu\,dx - \int_\alpha^\beta auv\,dx$$

$$= -\int_\alpha^\beta v''u\,dx + \int_\alpha^\beta u''v\,dx$$

Int. by Parts $\quad = -uv'\Big|_\alpha^\beta + \cancel{\int v'u'\,dx} + vu'\Big|_\alpha^\beta - \cancel{\int v'u'\,dx}$ ✓

b) $u(x) > 0$ in $\alpha < x < \beta$

$a(x) < b(x)$

$$\Rightarrow \int_\alpha^\beta \underbrace{(b-a)u}_{>0} v\,dx = \left[vu' - uv'\right]\Big|_\alpha^\beta$$

$$= \left[v(\beta)u'(\beta) - \cancel{u(\beta)}^{0} v'(\beta) - v(\alpha)u'(\alpha) + \cancel{u(\alpha)}^{0} v'(\alpha)\right]$$

$$= \left[v(\beta)u'(\beta) - v(\alpha)u'(\alpha)\right]$$

$u'(\beta) < 0 \qquad u'(\alpha) > 0$

$\therefore$ if $v < 0$, LHS < 0 ~~but~~ RHS > 0.

if $v > 0$, LHS > 0 but RHS < 0.

$\therefore$ v must have a 0!

c) $a(x) = b(x)$

$u'' + au = 0 \quad v'' + av = 0$

$$0 = \left[vu' - uv'\right]\Big|_{\alpha}^{\beta}$$ Let α, β be consecutive 0's of u.

$$= v(\beta)\underbrace{u'(\beta)}_{<0} - v(\alpha)\underbrace{u'(\alpha)}_{>0}$$

$\therefore v(\beta)$ and $v(\alpha)$ have opposite signs.

Repeat w/ γ, δ consecutive 0's of v to obtain desired result.

d) $b(x) \geq c^2$

$u'' + c^2 u = 0$ has solution $A\sin(cx) + B\cos(cx)$ (infinite zeros).

By part b), v has a zero between each consecutive zero, hence, infinitely many.

e) $v'' + \left(1 - \frac{1}{x^2}\right)v = 0$

$$1 - \frac{1}{x^2} \geq \frac{3}{4} \text{ for } x > 2$$

$\therefore$ by d) infinitely many zeros in this region.