HW 12

1) p.277 #2

$$Y_\ell^m(\theta,\varphi) = P_\ell^{|m|}(\cos\theta)e^{im\varphi}$$

Eigenfunctions of (3) + (9)

$$\begin{cases} \frac{1}{\sin^2\theta} Y_{\phi\phi} + \frac{1}{\sin\theta}(\sin\theta\, Y_\theta)_\theta + \gamma Y = 0 \\ Y(\theta,\varphi) \quad \text{period of } 2\pi \text{ in } \varphi \\ \qquad\qquad \text{finite at } \theta = 0, \pi \end{cases}$$

$$P_\ell^m(s) = \frac{(-1)^m}{2^\ell \ell!}(1-s^2)^{m/2} \frac{d^{\ell+m}}{ds^{\ell+m}}\left[(s^2+1)^\ell\right]$$

$\therefore \ell=0,\ m=0$

$$Y_0^0 = 1$$

$\therefore \ell=1,\ m=0$

$$P_1^0(s) = \frac{1}{2}\frac{d}{ds}\left[(s^2-1)\right] = s$$

$$\therefore Y_1^0 = \cos\theta$$

$\therefore \ell=1 \quad m=\pm 1$

$$P_1^1(s) = \frac{-1}{2}\sqrt{1-s^2}\,\frac{d^2}{ds^2}(s^2+1)$$

$$= -\sqrt{1-s^2}$$

$$\therefore Y_1^{\pm 1} = -\sqrt{1-\cos^2\theta}\,e^{i\varphi} = \underbrace{\sin\theta\cos\varphi}\ -\ i\underbrace{\sin\theta\sin\varphi}$$

rest similar...

2) p. 278 #10

$\{r > a\}$

$\frac{\partial u}{\partial r} = -\cos\Theta$ on $r = a$, bounded at ∞.

$R(r) = r^{\alpha}$ where $\alpha^2 + \alpha - \gamma = 0$

$Y = Y_\ell^m(\Theta, \varphi)$ with $\gamma = \ell(\ell+1)$

$0 = (\alpha - \ell)(\alpha + \ell + 1)$

Reject $\alpha = \ell$ as want finite as $\to \infty$.

so $\alpha = -\ell - 1$.

$$\Rightarrow u = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_{\ell m} r^{-\ell-1} P_\ell^m(\cos\Theta) e^{im\varphi}.$$

Want $\frac{\partial u}{\partial r} = -\cos\Theta$ on $r = a$

$$\frac{\partial u}{\partial r} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_{\ell m} (-\ell-1) a^{-\ell-2} P_\ell^m(\cos\Theta) e^{im\varphi} = -\cos\Theta$$

$$P_1^0 = \cos\Theta$$

$$\Rightarrow A_{10}(-2) a^{-3} \cos\Theta = -\cos\Theta$$

$$A_{10} = \frac{a^3}{2}.$$ all other $A_{\ell m} = 0$

$$\Rightarrow u = \frac{a^3}{2r^2}\cos\Theta + C$$ ← as bd. condition in u_r so arbitrary.

□

3) p.281 #1.

$V_{nm} = \sin(nx)\sin(my)$

		Multiplicity
$\lambda = 2$	$n=1 \quad m=1$	(1)
$\lambda = 5$	$(n=1 \quad m=2)$ or $(n=2 \quad m=1)$	(2)
$\lambda = 8$	$(n=2 \quad m=2)$	(1)
$\lambda = 10$	$(n=1 \quad m=3)$ or $(n=3 \quad m=1)$	(2)
$\lambda = 13$	$(n=2 \quad m=3)$ or $(n=3 \quad m=2)$	(2)
$\lambda = 17$	$(n=1 \quad m=4)$ or $(n=4 \quad m=1)$	(2)
$\lambda = 18$	$(n=3 \quad m=3)$	(1)
$\lambda = 20$	$(n=2 \quad m=4)$ or $(n=4 \quad m=2)$	(2)
$\lambda = 25$	$(n=3 \quad m=4)$ or $(n=4 \quad m=3)$	(2)

4) $\hat{x} = cx \qquad \hat{y} = cy$

λ_n, v_n eigen

a) $v|_{\partial\Omega} = 0$

if $(\bar{x}, \bar{y}) \in \partial\Omega \quad v(\bar{x}, \bar{y}) = 0$.

$\Rightarrow \hat{v}(c\bar{x}, c\bar{y}) = 0$

$\Rightarrow \hat{v}_n(x, y) = v_n(cx, cy)$.

$$
\begin{aligned}
-\Delta\hat{v} = -\Delta v(cx, cy) &= -c^2(v_{xx} + v_{yy}) \\
&= c^2 \lambda v(cx, cy) \\
&= c^2 \lambda \hat{v}
\end{aligned}
$$

$\Rightarrow \quad \hat{\lambda}_n = c^2 \lambda_n$

b) same as a).

5) p. 304 # 1

$F(0) = f(3) = 0$

$\int_0^3 [f(x)]^2 dx = 1$ and $\int_0^3 [f'(x)]^2 dx = 1$

$$\left\{ \frac{\|g'\|^2}{\|g\|^2} : g = 0 \text{ on } \{0,3\} : g \neq 0 \right\}.$$

is the minimum of $-\frac{d^2}{dx^2}$ on $[0,3]$ with $\|g\| = 1$

$-\frac{d^2}{dx^2} g = \lambda g$ $\qquad g = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$

$g(0) = 0$

so $A = 0$

$g = B\sin(\sqrt{\lambda}x)$

$g(3) = B\sin(\sqrt{\lambda}\,3) = 0$

so $\sqrt{\lambda} = \frac{n\pi}{3}$

$\therefore \lambda = \left(\frac{n\pi}{3}\right)^2$ smallest one is $\frac{\pi^2}{9} > 1$ so impossible to find f.

as $\frac{\|f'\|^2}{\|f\|^2} = 1.$

$\square$

6) p.304 #4

let $-\Delta v_j = \lambda_j v_j$ where $j \geq n$.

$$m^* \leq \frac{\int |\nabla v_j|^2}{\int v_j^2} = \frac{\int (-\Delta v_j)(v_j)}{\int v_j^2}$$

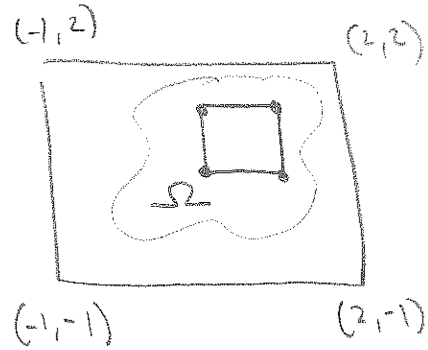
$$= \frac{\int \lambda_j v_j^2}{\int v_j^2} = \lambda_j$$

$\therefore$ $m^* \leq \lambda_j$ for $j \geq n$

as m^* is an eigenvalue (is $-\Delta v = m^* v$)

$m^* = \lambda_n$.

7)



Eigenfunctions on small rectangle are:

$$V_{nm}(x,y) = \sin(n\pi x)\sin(m\pi y)$$

with $\lambda_{nm} = (n^2+m^2)\pi^2$

Smallest eigenvalue is $2\pi^2$

On large rectangle.

$$V_{nm}(x,y) = \underbrace{\left(\cos(\beta(x+1)) + \sin(\beta(x+1))\right)}_{X(x)}\underbrace{\left(\cos(\gamma(y+1)) + \sin(\gamma(y+1))\right)}_{Y(y)}$$

$$X(2) = X(-1) = 0$$

$$X(-1) = A\cos(0) + B\sin(0) \Rightarrow A = 0.$$

$$X(x) = B\sin(\beta(x+1))$$

$$X(2) = B\sin(3\beta) \Rightarrow \beta = \frac{n\pi}{3}$$

$$\therefore V_{nm}(x,y) = \sin\left(\frac{n\pi x}{3}\right)\sin\left(\frac{m\pi y}{3}\right)$$

$$\therefore \lambda_{nm} = (n^2+m^2)\frac{\pi^2}{9}$$

$\therefore$ smallest is $\frac{2\pi^2}{9}$.

$\therefore$ eigenvalue is between $m = \frac{2\pi^2}{9}$ and $M = 2\pi^2$

$\square$

8) p 309 # 1

$-u'' = \lambda u$ in $(0,1)$ $u(0) = u(1) = 0$

$w_1 = x - x^2$ $w_2 = x^2 - x^3$

$w_1' = 1 - 2x$ $w_2' = 2x - 3x^2$

$$A_{ij} = \int_0^1 w_i' w_j' \, dx$$

$$B_{ij} = \int w_i w_j \, dx$$

$$\det(A - \lambda B) = 0$$

$$\left.\begin{aligned} a_{11} &= \int_0^1 (1-2x)^2 dx = 1/3 \\ a_{12} &= \int_0^1 (1-2x)(2x-3x^2) dx = 1/6 \\ a_{22} &= \int_0^1 (2x-3x^2)^2 dx = 2/5 \end{aligned}\right\} \quad A = \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 2/15 \end{bmatrix}$$

Likewise, $B = \begin{bmatrix} 1/30 & 1/60 \\ 1/60 & 1/105 \end{bmatrix}$

$$\det(A - \lambda B) = \left(1/3 - 1/30 \lambda\right)\left(2/15 - 1/105 \lambda\right) - \left(1/6 - 1/60 \lambda\right)^2 = 0$$

$\Rightarrow \lambda = 10, 42$ $\Leftarrow$ approximations

Actual are $\lambda = \pi^2 \approx 9.87$

$= \cancel{3}4\pi^2 \approx 39.48$

9)

a) A is real symmetric. λ_{max} largest eigenvalue.

$$A = QDQ^T \quad \text{where } D \text{ diagonal.}$$

$$c^T A c = c^T Q D Q^T c$$

let $b = Q^T c$

$$\Rightarrow c^T A c = b^T D b = \lambda_{max} b_1^2 + \dots + \lambda_{min} b_n^2$$

$$\leq \lambda_{max}(b_1^2 + \dots + b_n^2)$$

$$= \lambda_{max} b^T b$$

but as $Q^T Q = I$, $\quad b^T b = c^T Q^T Q c$

$$= c^T c$$

$$\Rightarrow c^T A c \leq \lambda_{max} c^T c.$$

$$\Rightarrow \lambda_{max} \geq \frac{c^T A c}{c^T c}.$$

Show λ_{max} attains this... let v_n be eigenvalue

$$\frac{v_n^T A v_n}{v_n^T v_n} = \frac{v_n^T \lambda_{max} v_n}{v_n^T v_n} = \lambda_{max} \checkmark.$$

$$\Rightarrow \lambda_{max} = \max_{c \neq 0} \frac{c^T A c}{c^T c} \quad \square$$

10) p 313 #2

$F(x)$ $\qquad$ $g(x)$ on ∂D.

$$\min_{w \in C^2} \frac{1}{2}\iiint_D |\nabla w|^2 dx - \iiint_D F w \, dx - \iint_{\partial D} g w \, dS.$$

$$\text{let } \iiint_D f \, dx + \iint_{\partial D} g \, dS = 0.$$

$$\frac{1}{2}\iiint_D |\nabla w|^2 = \iint_{\partial D} w \frac{\partial w}{\partial n} dS - \iiint_D w \Delta w \quad \text{by (G1).}$$

Let u be minimum and w any other function.

$$h(s) = \frac{1}{2}\int_D |\nabla u + s\nabla v|^2 dx - \int_D F(u+sv)\, dx - \int_{\partial D} g(u+sv)\, dS.$$

h is minimized at $s=0$. (as u minimum).

$$h'(s) = \int_D \nabla v \cdot (\nabla u + s\nabla v) - \int_D fv - \int_{\partial D} gv$$

$h'(0)=0$

$$\Rightarrow 0 = \int_D \nabla v \cdot \nabla u - \int fv - \int gv$$

$$= -\int v\Delta u + \int_{\partial D} v \frac{\partial u}{\partial n} - \int_D fv - \int_{\partial D} gv \quad (G1)$$

$$= -\int_D (\Delta u + f)v + \int_{\partial D} \left(\frac{\partial u}{\partial n} - g\right) v$$

as v arbitrary,

$$\Delta u = -f$$

$$\frac{\partial u}{\partial n} = g$$

11)

$$\frac{1}{2}\iiint_D |\nabla w|^2 dx - \iiint f w \, dx$$

let u minimize (same as 10)

$$h(s) = \frac{1}{2}\int_D |\nabla u + s\nabla v|^2 - \int f(u+sv)$$

$$h'(s) = \int \nabla v \cdot (\nabla u + s\nabla v) - \int_D f v$$

$$h'(0) = \int \nabla u \cdot \nabla v - \int_D f v$$

$$= -\int (\Delta u + f) v + \int$$

$\Rightarrow$ as v arbitrary $\Delta u = -f$

$\square$