Math 425, Spring 2015 **Jerry L. Kazdan**

Problem Set 4

DUE: Thurs. Feb. 12 in class. [Late papers will be accepted until 1:00 PM Friday.]

This week. Please read all of Chapter 3 and Chapter 4 in the Strauss text.

1. [One goal of this problem is to understand "periodic boundary conditions". See Part (e) below.]
   Say a function $u(t)$ satisfies the differential equation
   $$u'' + b(t)u' + c(t)u = 0 \tag{1}$$
   on the interval $[0,\, A]$ and that the coefficients $b(t)$ and $c(t)$ are both bounded, say $|b(t)| \le M$ and $|c(t)| \le M$ (if the coefficients are continuous, this is always true for some M).

   a) Define $E(t) := \frac{1}{2}(u'^2 + u^2)$. Show that for some constant γ (depending on M) we have $E'(t) \le \gamma E(t)$. [SUGGESTION: use the simple inequality $2xy \le x^2 + y^2$.]

   b) Show that $E(t) \le e^{\gamma t} E(0)$ for all $t \in [0,\, A]$. [HINT: First use the previous part to show that $(e^{-\gamma t}E(t))' \le 0$].

   c) In particular, if $u(0) = 0$ and $u'(0) = 0$, show that $E(t) = 0$ and hence $u(t) = 0$ for all $t \in [0,\, A]$. In other words, if $u'' + b(t)u' + c(t)u = 0$ on the interval $[0,\, A]$ and that the functions $b(t)$ and $c(t)$ are both bounded, and if $u(0) = 0$ and $u'(0) = 0$, then the only possibility is that $u(t) \equiv 0$ for all $t \ge 0$.

   d) Use this to prove the *uniqueness theorem*: if $v(t)$ and $w(t)$ both satisfy equation
   $$u'' + b(t)u' + c(t)u = f(t) \tag{2}$$
   and have the same initial conditions, $v(0) = w(0)$ and $v'(0) = w'(0)$, then $v(t) \equiv w(t)$ in the interval $[0,\, A]$.

   e) Assume the coefficients $b(t)$, $c(t)$, and $f(t)$ in equation (2) are periodic with period P, that is, $b(t + P) = b(t)$ etc. for all real t. If $\phi(t)$ is a solution of equation (2) that satisfies the *periodic boundary conditions*
   $$\phi(P) = \phi(0) \quad \text{and} \quad \phi'(P) = \phi'(0), \tag{3}$$
   show that $\phi(t)$ is periodic with period P: $\phi(t + P) = \phi(t)$ for all $t \ge 0$. Thus, the periodic boundary conditions (3) do imply the desired periodicity of the solution

2. Let C be a circle of radius 1 and let $u(\theta, t)$ be the temperature at a point θ, at time t. To make this well-defined, we need that $u(\theta + 2\pi) = u(\theta)$. Say $u(\theta, t)$ satisfies the heat equation $u_t = u_{\theta\theta}$. Let
   $$E(t) = \frac{1}{2}\int_{-\pi}^{\pi} u^2(\theta, t)\, d\theta.$$

a) Show that $E'(t) \le 0$.

b) If the initial temperature $u(\theta, 0) = 0$, show that $u(\theta, t) = 0$ for all $t \ge 0$.

3. Let $u(x,t)$ be the temperature at time t at the point x, $-L \le x \le L$. Assume it satisfies the heat equation $u_t = u_{xx}$ for $0 < t < \infty$ with the boundary condition $u(-L,t) = u(L,t) = 0$ and initial condition $u(x,0) = f(x)$.

   a) Show that $E(t) := \dfrac{1}{2}\displaystyle\int_{-L}^{L} u^2(x,t)\,dx$ is a decreasing function of t.

   b) Use this to prove uniqueness for the heat equation with these specified initial and boundary conditions $u(-L,t) = f(t)$, $u(L,t) = g(t)$.

   c) If $u(x,0) = \varphi(x)$ is an even function of x, and $u(-L,t) = u(L,t)$ show that the temperature $u(x,t)$ at later times is also an even function of x.

4. Find a formula for the solution of

$$u_{xx} + u_{xt} - 20u_{tt} = 0 \qquad \text{with} \qquad u(x,0) = \varphi(x) \;\text{ and }\; u_t(x,0) = \psi(x).$$

   Your result should be something like equation (8) on page 36 of the text.

5. [The dulcimer, P. 38 #5] Solve the wave equation $u_{tt} = c^2 u_{xx}$ with initial conditions $u(x,0) = 0$ and $u_t(x,0) = g(x)$, where $g(x) = 1$ if $|x| < a$ and $g(x) = 0$ for $|x| \ge a$. This corresponds to hitting the string with a hammer of width $2a$. Draw sketches of snapshots of the string (i.e., plot u versus x) for $t = \frac{1}{2}a/c$, $t = a/c$, $t = \frac{3}{2}a/c$, $t = 2a/c$ and $t = \frac{5}{2}a/c$. [Remark: This is very similar to p. 38 #3].

6. Solve the wave equation $u_{tt} = c^2 u_{xx}$ for the semi-infinite string $x \ge 0$ with the initial and boundary conditions

$$u(x,0) = 3 - \sin x, \quad u_t(x,0) = 0, \quad u(0,t) = 3 - t^2.$$

7. p. 60 #1

8. p. 60 #4

9. [p. 66 #1] Find an explicit formula for the solution of the wave equation $u_{tt} = u_{xx}$ on the half-line $0 < x < \infty$ with initial conditions $u(x,0) = \varphi(x)$, $u_t(x,0) = \psi(x)$ and the homogeneous Neumann boundary condition $u_x(0,t) = 0$.

10. p. 66 #3

[Last revised: June 28, 2015]