Problem Set 6

DUE: Thurs. Feb. 26 in class. [Late papers will be accepted until 1:00 PM Friday.]

This week. In Strauss please read Chapter 5 and the beginning of Chapter 6.

1. a) Write e^x as the sum of an even and an odd function of x.

   b) More generally, write any function $f(x)$ as the sum of an even function, $\varphi(x)$ and and odd function, $\psi(x)$, so $f(x) = \varphi(x) + \psi(x)$. Your goal is to find simple formulas for φ and ψ in terms of $f(x)$ and $f(-x)$.

2. Let $f(x) = \begin{cases} -1 & \text{for } -\pi < x < 0 \\ \ \ 1 & \text{for } \ \ 0 < x < \pi, \end{cases}$ $\quad g(x) = \begin{cases} 0 & \text{for } -\pi < x < 0 \\ 1 & \text{for } \ \ 0 < x < \pi, \end{cases}$

   and $h(x) = \begin{cases} 0 & \text{for } -\pi < x < -\pi/2 \\ 1 & \text{for } -\pi/2 < x < \pi/2 \\ 0 & \text{for } \pi/2 < x < \pi \end{cases}$

   How are the Fourier series for f and g related? Same question for $h(x)$.

   [In the above, all of the functions should first be extended so that they are 2π periodic.]

3. Let $f(x)$ be a 2π periodic function with (complex) Fourier series $f(x) = \sum_{-\infty}^{\infty} a_n e^{inx}$ and let α be a constant. We seek a 2π periodic solution of

   $$u'' + \alpha u = f(x)$$

   as a Fourier series $u(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$.

   a) Formally find the desired coefficients c_n in terms of the Fourier coefficients a_n of f.

   b) Are there any values of α for which there are difficulties? Explain.

4. Solve the wave equation $u_{tt} = c^2 u_{xx}$ for a string on $0 \le x \le L$ with fixed ends $u(0,t) = u(L,t) = 0$ assuming that initially it is plucked at its mid-point:

   $$u(x,0) = 1 - \frac{|x - (L/2)|}{L/2}, \qquad \text{and} \qquad u_t(x,0) = 0.$$

5. Consider the Fourier series $\displaystyle\sum_{-\infty}^{\infty} c_n e^{inx}$.

   a) If $|c_n| < \frac{Q}{n^2}$, where Q is a constant, show that the Fourier series converges absolutely.

b) If $\{c_n\}$ are the Fourier coefficients of f on $-\pi \leq x \leq \pi$ and if $|f(x)| \leq M$, show that $|c_n| \leq M$.

c) Assume $f \in C^1([-\pi, \pi])$ (that is, the first derivatives exist and are continuous) and f is 2π periodic, find a relationship between the Fourier coefficients c_n of f and γ_n of f' [HINT: Integrate by parts.]

In this case, conclude that $|c_n| \leq \text{const.}/n$.

MORAL: The smoother a function is the faster its Fourier coefficients decay.

6. [BESSEL'S INEQUALITY]

a) Let V_1, ..,V_N be orthogonal vectors and say

$$W = \sum_{k=1}^{N} c_k V_k + Z,$$

where Z is orthogonal to the V_k. Show that (Pythagoras)

$$\|W\|^2 = |c_1|^2\|V_1\|^2 + \cdots + |c_N|^2\|V_N\|^2 + \|Z\|^2.$$

In particular,

$$\sum_{k=1}^{N} |c_k|^2\|V_k\|^2 \leq \|W\|^2.$$

b) Say on $-\pi \leq x \leq \pi$ the formal Fourier series for f is $\sum_{-\infty}^{\infty} c_k e^{ikx}$. Show that for any N

$$\tfrac{1}{2\pi}\|f\|^2 \geq \sum_{k=-N}^{N} |c_k|^2.$$

In particular, $\sum_{k=-\infty}^{\infty} |c_k|^2$ converges.

7. [Strauss, p. 117 #5] Show that the Fourier sine series on $(0, \ell)$ can be derived from the full Fourier series on $(-\ell, \ell)$ as follows.

Let $\phi(x)$ be any continuous function on $(0, \ell)$. Let $\tilde{\phi}$ be its odd extension. Write the full series for $\tilde{\phi}$ on $(-\ell, \ell)$ [assume that its sum is indeed $\tilde{\phi}$]. Use that if ϕ is an odd function then its full Fourier series only has sine terms. Now simply restrict your attention to $0 < x < \ell$ to find the sine series for $\phi(x)$.

8. [Strauss p. 117 #8] Prove that differentiation switches an odd function to an even function, and an even function to an odd function. [There is a similar assertion for integration.]

9. [Strauss p. 118 #15] Without any computation, predict which of the Fourier coefficients of $|\sin x|$ on the interval $(-\pi, \pi)$ must vanish.

10. Say a real-valued function $f(x)$ has the complex Fourier series $\sum_{-\infty}^{\infty} c_n e^{inx}$. Show that $c_{-n} = \overline{c_n}$.

    [The converse is also true: If $c_{-n} = \overline{c_n}$, then the function is real-valued.]

[Last revised: March 17, 2015]