1) a) $\int_{-1}^{1} cx\,dx = c\frac{x^2}{2}\Big|_{-1}^{1} = 0$

b) $\int_{-1}^{1} (ax^2+bx+c)\,dx = \frac{a}{3}x^3 + \frac{b}{2}x^2 + cx\Big|_{-1}^{1} = \frac{2a}{3} + 2c$

$\int_{-1}^{1} ax^3+bx^2+cx\,dx = \frac{a}{4}x^4 + \frac{b}{3}x^3 + \frac{c}{2}x^2 = \frac{2b}{3} \Rightarrow b = 0$

$a=1 \qquad c = -1/3$

$f(x) = x^2 - 1/3$

c) $\int_{-1}^{1} x^3+bx^2+cx+d\,dx = \frac{1}{4}x^4 + \frac{b}{3}x^3 + \frac{c}{2}x^2 + dx\Big|_{-1}^{1} = \frac{2b}{3} + 2d = 0$

$\int_{-1}^{1} x^4+bx^3+cx^2+dx\,dx = \frac{2}{5} + \frac{2c}{3} = 0 \Rightarrow c = -\frac{3}{5}$

$\int_{-1}^{1} x^5+bx^4+cx^3+dx^2\,dx = \frac{2b}{5} + \frac{2d}{3} = 0$

$\begin{cases} b+3d=0 \\ 3b+5d=0 \end{cases} \Rightarrow b=d=0$

$f(x) = x^3 - \frac{3}{5}x$

(If Normalized are Legendre polynomials).

2) $u_{tt} = c^2 u_{xx}$ on $(0, \ell)$

$u(0,t) = 0$ $\quad$ $u(x,0) = x$

$u_x(\ell, t) = 0$ $\quad$ $u_t(x,0) = 0$

$$X''(x) + \beta^2 X(x) = 0$$

$$X(x) = A\cos(\beta x) + B\sin(\beta x)$$

$$X(0) = 0 \Rightarrow A = 0$$

$$X'(\ell) = 0 \Rightarrow B\cos(\beta \ell) = 0$$

$$\Rightarrow \beta \ell = \left(\tfrac{1}{2} + n\right)\pi$$

$$\beta_n = \frac{(\frac{1}{2}+n)\pi}{\ell}$$

$$\Rightarrow X_n(x) = B_n \sin\left(\frac{(\frac{1}{2}+n)\pi x}{\ell}\right)$$

$$T'' + c^2\beta^2 T = 0$$

$$T(t) = C\cos(\beta c t) + D\sin(\beta c t)$$

$$T'(0) = 0$$

$$\Rightarrow D = 0$$

$$\Rightarrow T_n(t) = C_n \cos\left(\frac{(\frac{1}{2}+n)c\pi t}{\ell}\right)$$

Note that $T_n(0) = C_n$

Therefore $$u(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{(\frac{1}{2}+n)c\pi}{\ell} t\right) \sin\left(\frac{(\frac{1}{2}+n)\pi}{\ell} x\right)$$

Needs to satisfy $u(x,0) = x$

$$\Rightarrow \quad x = u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{(\frac{1}{2}+n)\pi}{\ell} x\right)$$

We know $A_n = \dfrac{\langle x, X_n \rangle}{\langle X_n, X_n \rangle}$ where $X_n = \sin\left(\frac{(\frac{1}{2}+n)\pi}{\ell} x\right)$

$$\langle x, X_n \rangle = \int_0^{\ell} x \sin\left(\frac{(\frac{1}{2}+n)\pi}{\ell} x\right) dx \qquad \begin{array}{ll} U = x & dV = \sin(\quad)\,dx \\ dU = dx & V = -\frac{\ell}{(\frac{1}{2}+n)\pi}\cos(\quad) \end{array}$$

$$= \frac{(-1)^n \ell^2}{\pi^2 (n+\frac{1}{2})^2}$$

$$\langle X_n, X_n \rangle = \int_0^{\ell} \sin^2(\qquad)\,dx = \frac{2}{\ell}$$

$$\Rightarrow A_n = \frac{2\ell(-1)^n}{\pi^2 (n+\frac{1}{2})^2}$$

3) Symmetric $\iff f'(x)g(x) - f(x)g'(x)\Big|_a^b = 0$

$$\iff f'(b)g(b) - f(b)g'(b) - f(a)g(a) - f(a)g'(a) = 0$$

$$\iff [\gamma f(a) + \delta f'(a)](\alpha g(a) + \beta g'(a))$$

$$- (\alpha f(a) + \beta f'(a))(\gamma g(a) + \delta g'(a))$$

$$- f'(a)g(a) + f(a)g'(a) = 0$$

$$\iff (\alpha\delta - \beta\gamma - 1)(f'(a)g(a) - f(a)g'(a)) = 0$$

$$\iff \alpha\delta - \beta\gamma - 1 = 0$$

4) We have $X''(x) = \lambda X(x)$

$$\int_a^b X''(x)X(x)\,dx = -\underbrace{\int_a^b [X'(x)]^2\,dx}_{\text{less than zero}} + \underbrace{X'(x)X(x)\Big|_a^b}_{\substack{\text{less than zero} \\ \text{by assumption}}}$$

$$\parallel$$

$$-\lambda \int_a^b [X(x)]^2\,dx$$

$$\Rightarrow \lambda \geq 0.$$

5) $X^{(4)} = \lambda X$

$$\int_0^\ell X^{(4)} X = -\int_0^\ell X''' X' \, dx + \cancel{X''' X \Big|_0^\ell}^{\text{0 by bd condition}} = \underbrace{\int_0^\ell (X'')^2 \, dx}_{\geq 0} - \cancel{X'' X \Big|_0^\ell}^{\text{0 by bd. condition}}$$

$$\parallel$$

$$\lambda \int_0^\ell X^2 \, dx$$

$\lambda \geq 0$.

6) a) $E'(t) = \int_0^\ell u_t u_{tt} + u_x u_{xt} \, dx$

$$= \int_0^\ell u_t u_{xx} + u_x u_{xt} \, dx$$

$$= -\int_0^\ell u_x u_{tx} \, dx + \cancel{u_x u_{tx} \Big|_0^\ell}^{\text{0 by bd.}} + \int u_x u_{xt} \, dx \text{ by Integration by Parts}$$

$$= 0.$$

b) $u = \sum T_n(t) X_n(x)$

$$\Rightarrow E_n(t) = \frac{1}{2} \int_0^\ell (T'X)^2 + (TX')^2 \, dx$$

$u_t = \sum_n v_n(t) X_n(x)$

where $v_n(t) = \frac{2}{\ell} \int_0^\ell u_t X_n(x) \, dx = \frac{\partial}{\partial t} \frac{2}{\ell} \int_0^\ell u X_n = T_n'(t)$

Likewise

$$u_x = \sum_n T_n X_n'.$$

$$\Rightarrow E(t) = \frac{1}{2} \int_0^\ell \left[\sum T_n' X_n\right]^2 + \left[\sum_n T_n X_n'\right]^2 dx$$

by orthogonality

$$= \frac{1}{2} \int_0^\ell \sum (T_n' X_n)^2 + (T_n X_n')^2 \, dx = \sum_n E_n$$

7) $0 \le \|f+tg\|^2 = \langle f+tg, f+tg\rangle = \|f\|^2 + 2t\langle f,g\rangle + t^2\|g\|^2$

minimized when $t = -\dfrac{\langle f,g\rangle^2}{\|g\|^2}$

$$\Rightarrow \|f\|^2 - \frac{\langle f,g\rangle^2}{\|g\|^2} \ge 0$$

$$\Rightarrow |\langle f,g\rangle| \le \|f\|\,\|g\|$$

8) $u_t = u_{xx}$ in $(0,\ell)$

$u_x(0,t) = 0$

$u(1,t) = 1$

$u(x,0) = x^2$

As this is Neumann, expect cosine series!

$\left(-X'' = \lambda X \quad \text{with } X'(0) = 0 \Rightarrow \text{no sines}\right)$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} u_n(t) \cos\left((n+\tfrac{1}{2})\pi x\right)$$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} v_n(t) \cos\left((n+\tfrac{1}{2})\pi x\right)$$

where

$$v_n(t) = 2\int_0^1 u_t \cos\left((n+\tfrac{1}{2})\pi x\right) = \frac{du_n}{dt}$$

$$\frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty} w_n(t) \cos\left((n+\tfrac{1}{2})\pi x\right)$$

where $w_n(t) = 2\int_0^1 u_{xx} \cos\left((n+\tfrac{1}{2})\pi x\right) dx$

$$= -2\int_0^1 ((n+\tfrac{1}{2})\pi)^2 u \cos((n+\tfrac{1}{2})\pi x) + 2\left[n\pi u \sin((n+\tfrac{1}{2})\pi x) + u_x \cos((n+\tfrac{1}{2})\pi x)\right]\Big|_0^1$$

by Green's 2^{nd} identity.

$$= -\lambda_n u_n(t) + 2\left[(n+\tfrac{1}{2})\pi u(1,t)\sin((n+\tfrac{1}{2})\pi x) - u_x(0,t)\right]$$

where $\lambda_n = ((n+\frac{1}{2})\pi)^2$

$$= -\lambda_n u_n(t) + 2\left[(-1)^n (n+\tfrac{1}{2})\pi\right]$$ as $u(1,t)=1$, $u_x(0,t)=0$

Now,

$$V_n(t) - W_n(t) = 2\int_0^1 (u_t - k u_{xx}) \cos((n+\tfrac{1}{2})\pi x)\,dx = 0$$

$$\Rightarrow \frac{du_n}{dt} = -\lambda_n u_n + \left[(-1)^n(2n+1)\pi\right]$$

$$\Rightarrow u_n(t) = Ce^{-\lambda_n t} + (2n+1)\pi\int_0^t e^{-\lambda_n(t-s)}(-1)^n\,ds$$

$$= C_n e^{-\left(\frac{n+1}{2}\right)^2\pi^2 t}$$

To solve C_n, use $u(x,0)=x^2$

$$x^2 = \sum_{n=1}^{\infty} C_n \cos((n+\tfrac{1}{2})\pi x)$$

$$C_n = 2\int_0^1 x^2\cos((n+\tfrac{1}{2})\pi x) = (-1)^{n+1} 4 (n+\tfrac{1}{2})^{-3}\pi^{-3}$$

b) only first term.

9) $u_{tt} = c^2 u_{xx} + g(x)\sin(\omega t)$

$$u(x,t) = \sum u_n(t) \sin\left(\frac{n\pi x}{\ell}\right)$$

$$\Rightarrow \quad \frac{d^2 u_n}{dt^2} + c^2 \lambda_n u_n = f_n(t)$$

where $\quad g(x)\sin(\omega t) = \sum_{n=1}^{\infty} f_n(t) \sin\left(\frac{n\pi x}{\ell}\right)$

$$f_n(t) = \frac{2}{\ell}\int_0^\ell g(x)\sin(\omega t)\sin\left(\frac{n\pi x}{\ell}\right) = A_n \sin(\omega t)$$

where $\quad A_n = \frac{2}{\ell}\int_0^\ell g(x)\sin\left(\frac{n\pi x}{\ell}\right)$

$= \hat{g}(n)$

$$\Rightarrow U_n = C_n^{(1)}\cos\left(\frac{n\pi c t}{\ell}\right) + C_n^{(2)}\sin\left(\frac{n\pi c t}{\ell}\right) + \frac{A_n}{c^2\lambda_n - \omega^2}\sin(\omega t)$$

$$U_n(0) = 0 = C_n^{(1)}$$

$$u = \sum_n \frac{A_n}{\omega^2 - c^2\lambda_n}\left(\frac{\omega \ell}{n\pi c}\sin\left(\frac{n\pi c t}{\ell}\right) - \sin(\omega t)\right)\sin\left(\frac{n\pi x}{\ell}\right)$$

10) p.160 #2

$$\Delta_3 u = u_{rr} + \frac{2}{r} u_r + \frac{1}{r^2}\left[u_{\theta\theta} + \cot(\theta) u_\theta + \frac{1}{\sin^2\theta} u_{\varphi\varphi}\right]$$

$$u_\theta = u_\varphi = 0$$

$$\Rightarrow \Delta_3 u = u_{rr} + \frac{2}{r} u_r = k^2 u$$

let $u = V/r$

$$u_r = \frac{1}{r} V_r - \frac{1}{r^2} V$$

$$u_{rr} = \frac{1}{r} V_{rr} - \frac{1}{r^2} V_r - \frac{1}{r^2} V_r + \frac{2}{r^3} V$$

$$\Rightarrow \frac{1}{r} V_{rr} \cancel{- \frac{2}{r^2} V_r} \cancel{+ \frac{2}{r^3} V} \cancel{+ \frac{2}{r^2} V_r} \cancel{- \frac{2}{r^3} V} = \frac{k^2 V}{r}$$

$$\Rightarrow V_{rr} = k^2 V$$

$$\Rightarrow V = C_1 e^{kr} + C_2 e^{-kr}$$

$$\Rightarrow u = \frac{V}{r}$$

$\square$

11) $\Delta u = 1$ in $a < r < b$

$u(a, \theta) = u(b, \theta) = 0$

Answer ind. of Θ. $\Rightarrow u(r, \theta) = u(r)$

$u_{rr} + \frac{1}{r} u_r = 1$

$u(a) = 0$

$u(b) = 0$

$u_{rr} + \frac{1}{r} u_r = 0$ has sol'n $C_1 \log r + C_2$

$u_{rr} + \frac{1}{r} u_r = 1$ has particular sol'n $\frac{r^2}{4}$

$$u = \frac{r^2}{4} + C_1 \log r + C_2$$

$\left.\begin{array}{l} u(a) = 0 \\ u(b) = 0 \end{array}\right\}$ $C_1 = \dfrac{b^2 - a^2}{4(\log a - \log b)}$ $C_2 = \dfrac{a^2 \log b - b^2 \log a}{4(\log a - \log b)}$

12) $w = u - v$

as $w = 0$ on bd as $\Delta w = 0$

$$E = \frac{1}{2}\int_D \nabla w \cdot \nabla w \, dx = \frac{1}{2}\oint_{\partial D} \cancel{w (\nabla w \cdot n)} ds - \frac{1}{2}\int_D \cancel{w \Delta w} \, dx$$

$\therefore \nabla w = 0$

$\Rightarrow$ uniqueness

13) If $f \geq g$

Max principle says $u \geq v$ as

$w = u - v$ has bd data $h = f - g \geq 0$

$\therefore w \geq 0$

$\therefore u - v \geq 0$

$\Rightarrow u \geq v$ $\square$

14) $$\iiint_D f = \iiint \Delta u = \oiint_{\partial D} (\nabla u \cdot n)\, dS = \oiint_{\partial D} g \, dS$$