Problem Set 8

DUE: Thurs. Mar. 26 in class. [Late papers will be accepted until 1:00 PM Friday.]

This week. Please read the Chapters 7 and 9 in the Strauss text.

[Lots of problems. Again, fortunately, most of them are short.]

1. Strauss, p. 172 #1

2. Strauss, p. 172 #2

3. Strauss, p. 175 #1

4. Solve $\Delta u = 0$ in the annulus $1 \le x^2 + y^2 \le 2$ with $u(x, y) = 1$ on the circle $x^2 + y^2 = 1$ and $u(x, y) = 7$ on $x^2 + y^2 = 2$.

5. Suppose u is a twice differentiable function on $\mathbb{R}$ which satisfies the ordinary differential equation
$$u'' + b(x)u' - c(x)u = 0,$$
where $b(x)$ and $c(x)$ are continuous functions on $\mathbb{R}$ with $c(x) > 0$ for every $x \in (0, 1)$.

   a) Show that u cannot have a positive local maximum in the interval $(0, 1)$, that is, have a local maximum at a point p where $u(p) > 0$. Also show that u cannot have a negative local minimum in $(0, 1)$.

      [The example $u'' + u = 0$ has $u(x) = \sin x$ as a solution, which does have posivive local maxima and negative local minima. This shows that some assumption, such as our $c(x) > 0$ is needed.]

   b) If $u(0) = u(1) = 0$, prove that $u(x) = 0$ for every $x \in [0, 1]$.

   c) If u satisfies
   $$4u_{xx} + 3u_{yy} - 5u = 0$$
   in a region $\mathcal{D} \subset \mathbb{R}^2$, show that it cannot have a local positive maximum. Also show that u cannot have a local negative minimum.

   d) Repeat the above for a solution of
   $$4u_{xx} - 2u_{xy} + 3u_{yy} + 7u_x + u_y - 5u = 0.$$
   [REMARK: If $A = (a_{ij})$ and $B = (b_{ij})$ are positive semi-definite symmetric $n \times n$ matrices, then $\sum_{i,j=1}^{n} a_{ij}b_{ij} \ge 0$.]

e) If a function $u(x, y)$ satisfies the above equation in a bounded region $\mathcal{D} \in \mathbb{R}^2$ and is zero on the boundary of the region, show that $u(x, y)$ is zero throughout the region.

6. Consider the Dirichlet problem $\Delta u - 5u = 0$ in a bounded region $\Omega \subset \mathbb{R}^2$ with $u(x, y) = f(x, y)$ for points (x, y) on the boundary $\partial\Omega$. Prove the uniqueness in two ways: using a maximum principle (see the previous problem) and using an energy argument.

7. a) Let B be the ball $\{r^2 = x^2 + y^2 + z^2 < a^2\}$ in $\mathbb{R}^3$. Compute all the *radial* eigenfunctions $u(r)$ of $-\Delta$ with Neumann boundary conditions $\partial u/\partial r = 0$ for $r = a$. Thus, you are solving $-[u_{rr} + \frac{2}{r} u_r] = \lambda u$. [SUGGESTION: the substitution $v(r) = ru(r)$ is useful. Note it implies $v(0) = 0$.]

   b) Compute the corresponding eigenvalues (there is an explicit formula).

   c) Use this to solve the heat equation $u_t = \Delta u$ in B with $u_r = 0$ on the boundary in the special case where the initial temperature, $u(x, 0) = \varphi(r)$ depends only on r. Your solution will be an infinite series. Please include a formula for finding the coefficients.

8. Strauss, p. 184 #2

9. Strauss, p. 184 #5

10. Strauss, p. 187 #1

11. Strauss, p. 187 #2

12. Strauss, p. 190 #2

[Last revised: March 21, 2015]