Math 609 Jerry L. Kazdan
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Complex Analysis Exam II

DIRECTIONS This exam has two parts, Part A has 4 short answer problems (5 points each so 20 points) while Part B has 7 traditional problems, 10 points each so 70 points).
Closed book but you may use one 3×5 card with notes (on both sides).
All contour integrals are assumed to be in the positive sense (counterclockwise).

Short Answer Problems [5 points each (20 points total)]

A1. If $f(z)$ is an entire function with $|f(z)| \ge 1$ everywhere, what can you conclude about f? Justify your assertions.

Solution Then $g(z) := 1/f(z)$ is an entire function. Moreover $|g(z)| \le 1$. Consequently g is a constant, so f is a constant.

A2. If $f(z)$ is an entire function and $f(x+2\pi) = f(x)$ for all real x, does $f(z+2\pi) = f(z)$ for all complex z? Proof or counterexample.

Solution True. Let $g(z) := f(z+2\pi) - f(z)$. then g is entire and vanishes on the real axis. Consequently $g(z) = 0$ everywhere.

A3. The function $\dfrac{z^3-1}{z^2+3z-4}$ has a power series expansion in a neighborhood of the origin. What is its radius of convergence? Justify your assertion.

Solution Since $z^2+3z-4 = (z-1)(z+4)$, the point $z=1$ is a removable singularity. Thus the radius of convergence is 4.

A4. Assume the entire function $f(z)$ has no zeroes on any of the circles $|z| = n$, $n = 1, 2, 3, \ldots$ and also that

$$\oint_{|z|=n} \frac{1}{f(z)}\,dz \ne \oint_{|z|=n+1} \frac{1}{f(z)}\,dz, \quad n = 1, 2, 3, \ldots.$$

Is this function transcendental? Proof or counterexample.

Solution The assumption implies that $f(z)$ has at least one zero in the annulus $n < |z| < n+1$. Thus it has infinitely many zeros so must be transcendental.

Traditional Problems [10 points each (70 points total)]

B1. Assume $f(z)$ is meromorphic for all $|z| < \infty$ and satisfies

$$|f(z)| \le \left(\frac{2|z|}{|z-1|}\right)^{3/2}.$$

What can you conclude about f? Justify your assertions.

Solution We claim the only possibility is $f(z) \equiv 0$.

PROOF: Clearly the only possible singularity of $f(z)$ is at $z = 1$. Let $g(z) := (z-1)^2 f(z)$. Then

$$|g(z)| \le (2|z|)^{3/2} |z-1|^{1/2}.$$

Since $g(z)$ is bounded near $z = 1$, it has at most a removable singularity there. Consequently g is an entire function. Because it grows at most like $|z|^2$ for large z, it must be a quadratic polynomial. But $g(z) := (z-1)^2 f(z)$ so $f(z) \equiv$ constant. Noticing that $g(0) = 0$ we conclude that $f(0) = 0$. Hence $f(z) \equiv 0$.

B2. Evaluate $A = \displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2}\, dx$ where $a > 0$.

Solution Consider $B_R := \displaystyle\oint_{\Gamma_R} \frac{e^{iz}}{z^2 + a^2}\, dz$ over the semicircle Γ_R which is the boundary of the half-disk $|z| = R$ in the upper-half plane $y > 0$. For $R > a$ the only singularity of the integrand inside Γ_R is at $z = ia$. Thus by the residue theorem $B_R = 2\pi i \dfrac{e^{-a}}{2ia} = \dfrac{\pi e^{-a}}{a}$.

But also

$$B_R = \int_{-R}^{R} \frac{e^{ix}}{x^2 + a^2}\, dx + \int_{\gamma_R} \frac{e^{iz}}{z^2 + a^2}\, dz = I_1 + I_2,$$

where γ_R is the semi-circle $z = Re^{i\theta}$ for $0 \le \theta \le \pi$. Since $|e^{iz}| = |e^{ix-y}| = e^{-y} \le 1$ on γ_R, then for R large $I_2 = O(1/R) \to 0$.

Letting $R \to \infty$ we conclude that

$$\frac{\pi e^{-a}}{a} = B_R \to \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 + a^2}\, dx.$$

. Taking the real parts of both sides, we conclude that

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2}\, dx = \frac{\pi e^{-a}}{a}.$$

B3. a) Let $f(z)$ be holomorphic in $|z| \le R$ with $|f(z)| \le M$ on $|z| = R$. Show that

$$|f(z) - f(0)| \le \frac{2M|z|}{R} \tag{1}$$

Solution To be brief, apply the Schwarz Lemma to $g(z) := f(z) - f(0)$ and note that $|g(z)| \le 2M$ on $|z| = R$.
In greater detail, since $g(z)/z$ is holomorphic in $|z| \le R$, by the maximum principle $|g(z)/z| \le 2M/R$.

b) Use this to give a proof of Liouville's theorem.

Solution Let $R \to \infty$ in (1).

B4. If $f(t)$ is piecewise continuous and uniformly bounded for all $t \ge 0$, show that for $Re\{z\} > 0$ the function (Laplace transform)

$$g(z) := \int_0^\infty f(t)e^{-zt}\, dt$$

is holomorphic for $\operatorname{Re}\{z\} > 0$.

Solution The piecewise continuity and boundedness of f imply that the improper integral exists for $Re\{z\} > 0$.

Step 1: Let $g_c(t) := \int_0^c f(t)e^{-zt}\, dt$. We claim that $g_c(z)$ is an entire function. We explicitly show that g_c has a complex derivative:

$$g_c'(z) = -\int_0^c f(t)e^{-zt} t\, dt.$$

Indeed, since $\dfrac{e^{-ht} - 1}{h} + t$ converges to 0 uniformly for $t \in [0, c]$, then

$$\frac{g_c(z+h) - g_c(z)}{h} - \left(-\int_0^c f(t)e^{-zt} t\, dt\right) = \int_0^c f(t)e^{-zt}\left[\frac{e^{-ht} - 1}{h} + t\right] dt \to 0.$$

Step 2: To complete the proof, we claim the entire functions $g_c(z)$ converge uniformly to $g(z)$ in the half-space $\operatorname{Re}\{z\} \ge \delta$ for any $\delta > 0$. Say $|f(t)| \le M$. Then

$$|g(z) - g_c(z)| = \int_c^\infty |f(t)e^{-zt}|\, dt \le \int_c^\infty Me^{-\delta t}\, dt = \frac{Me^{-\delta c}}{\delta}$$

which converges to zero as $c \to \infty$.

As alternates one can use Morera's Theorem (but justify ine interchange of the order of integration) or directly take the derivative of $g(z)$, justifying why one can differentiate under the integral (say by the dominated convergence theorem).

B5. Let $f_n(z)$ be a sequence of functions holomorphic in the connected open set Ω and assume they converge uniformly on every compact subset of Ω. Show that the sequence of derivatives $f_n'(z)$ also converges uniformly on every compact subset of Ω.

Solution Let $K \in \Omega$ be any compact set and let r be the distance from K to the boundary of Ω. There is a cover of K by a finite number of open disks of radius $r/4$. Say $|z-a| < r/4$ is one of these disks. Then by the Cauchy Integral Formula applied to the larger disk $|\zeta - a| < r/2$, for any point z in this smaller disk

$$f_n'(z) = \frac{1}{2\pi i} \oint_{|\zeta-a|<r/2} \frac{f_n(\zeta)}{(\zeta - z)^2}\, d\zeta$$

But since $|\zeta - z| > r/2$ and the f_n's converge uniformly, we can pass limit under the integral and conclude that the f_n's converge uniformly in this disk:

$$f_n'(z) \to \frac{1}{2\pi i} \oint_{|\zeta-a|<r/2} \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta = f'(z).$$

Because this finite collection of smaller disks cover K, we conclude that the convergence is uniform in K.

B6. Find a conformal map from the unbounded region outside the disks $\{|z+1| \le 1\} \cup \{|z-1| \le 1\}$ to the upper half plane.

Solution Step 1: Writing $w = u + iv$, the map $w = 1/z$ maps this region to the vertical strip $-1/2 < u < 1/2$. [In greater detail, it maps the real axis to itself, the imaginary axis to itself, the origin to infinity, and hence the circles $|z \pm 1| = 1$ to the vertical straight lines $u = \pm 1/2$.]

Step 2: By a translation, rotation, and stretching we can map this strip to the horiontal strip $-\infty < s < \infty$, $0 < t < \pi$ in the $\zeta = s + it$ plane.

Step 3: Then e^{ζ} maps this strip to the upper half-plane.

B7. Consider the family of polynomials

$$p(z; t) = z^n + a_{n-1}(t) z^{n-1} + \cdots a_1(t) z + a_0(t),$$

where the coefficients $a_j(t)$ depend continuously on the parameter $t \in [0, 1]$. Assume that at $t = 0$ the polynomial $p(z; 0)$ has k zeroes (counted with their multiplicity) in the disk $|z - c| < R$ and has no zeroes on the circle $|z - c| = R$.

Show that for all sufficiently small t the polynomial $p(z; t)$ also has k zeroes in $|z - c| < R$.

Solution Write

$$p(z; t) = p(z; 0) + [p(z; t) - p(z; 0)].$$

Since the circle $|z - c| = R$ is compact, on this circle $|p(z; 0)| \ge c$ for some $c > 0$. Now pick t so small that $|p(z; t) - p(z; 0)| < c/2$ in this disk. Then by Rouché's theorem $p(z; t)$ and $p(z; 0)$ both have the same number of zeroes in this disk.

Remark: Although we picked $t \in [0, 1]$, that was essentially irrellenvent.