THE GROUP C_n. BASIC PROPERTIES.

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1. Definitions and notations.

In the first lecture we learned a lot of things about the series of Coxeter groups S_n, $n \geq 1$. Here we start to study another series of Coxeter groups. The n-th group of this series is denoted by C_n, $n \geq 1$. We define it as an abstract group isomorphic to the group of all isometries of n-dimensional cube.

Exercise 1. Compute the order of C_n.

Hint. Consider the homogeneous space X_{2n} consisting of centers of $(n-1)$-dimensional faces and show that a stabilizer of a point is isomorphic to C_{n-1}.

Exercise 2. Show that C_n is isomorphic to the semidirect product of a subgroup S_n and a normal abelian subgroup $A_n \simeq (\mathbb{Z}/2\mathbb{Z})^n$. Give the geometric interpretations for elements of S_n and of A_n.

Theorem 1. The group C_n is a Coxeter group with the graph

$$\circ - - \circ - - \circ - - \cdots - - \circ - - \circ \overset{4}{- -} \circ \tag{1}$$

Proof. Choose an orthogonal coordinate system in $\mathbb{R}^n$ so that our cube is bounded by hyperplanes $H_k^{\pm} : x_k = \pm 1$. Let s_k, $1 \leq k \leq n-1$, be a reflection in $\mathbb{R}^n$ interchanging x_k and s_{k+1}. Introduce also a reflection s_n which change the sign of x_n and fixes all other coordinates. The theorem follows from

Exercise 3. Show that reflections s_k, $1 \leq k \leq n$, generate C_n and satisfy the Coxeter relations:

$$(s_k s_j)^{m_{i,j}} = e \quad \text{for} \quad m_{i,j} = \begin{cases} 1 & \text{if} \quad i = j \\ 2 & \text{if} \quad |i-j| > 1 \\ 3 & \text{if} \quad j = i+1 < n \\ 4 & \text{if} \quad j = i+1 = n. \end{cases} \tag{2}$$

The group C_n contains a subgroup isomorphic to S_n, but in its turn is contained in S_{2n}. Indeed, from Exercise 1 we see that $C_n/C_{n-1} \simeq X_{2n}$. Hence, $C_n \subset \mathrm{Aut} X_{2n} \simeq S_{2n}$.

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Exercise 4. Let $\sigma \in S_{2n}$ be a permutation with n cycles of length 2. Show that the centralizer of σ in S_2n is isomorphic to C_n.

Using Exercise 4 we can realize the subgroup $C_n \in S_{2n}$ as follows. Let X_{2n} be realized as the set of numbers $\pm 1, \pm 2, \ldots, \pm n$ and permutation σ acts as a multiplication by -1. Then elements of C_n are exactly those permutations $s \in S_{2n}$ which have the property

$$s(-k) = -s(k). \tag{3}$$

Let now $s \in C_n \subset S_{2n}$ and $\langle s \rangle$ be a cyclic subgroup generated by s. Consider the orbits of $\langle s \rangle$ in X_{2n}. The set of orbits is invariant under σ. But the orbits themselves are not necessarily invariant. More precisely, there are two kind of orbits:

1. The σ-invariant orbits $\Omega = -\Omega$ of length $2k$.
2. The pairs $\Omega, -\Omega$ of disjoint orbits of length l.

By an inner automorphism (i.e. renaming of numbers so, that if k goes to m, than $-k$ goes to $-m$), we can reduce the orbit of the first kind to the form

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots \longrightarrow k \longrightarrow -1 \longrightarrow -2 \longrightarrow \cdots \longrightarrow -k \longrightarrow 1.$$

A pair of orbits of the second kind can be reduced to the form

$$\begin{array}{c} 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots \longrightarrow l \longrightarrow 1 \\ -1 \longrightarrow -2 \longrightarrow -3 \longrightarrow \cdots \longrightarrow -l \longrightarrow -1 \end{array}$$

It is rather clear that the conjugacy class of s in C_n is determined by lengths of all cycles of the first kind and by lengths of all cycles of the second kind. Thus, as a label of the conjugacy class of s we can take a pair of partitions of the number n.

Namely, $\lambda = \{\lambda_1 \geq \lambda_2 \geq \ldots \lambda_p > 0\}$ are half-lengths of cycles of the first kind and $\mu = \{\mu_1 \geq \mu_2 \geq \ldots \mu_q > 0\}$ are the lengths of cycles of the second kind.

Exercise 5. Compute the cardinality of the class $C_{\lambda,\mu} \subset C_n$.

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