THE GROUP S_n. BASIC PROPERTIES.

A. A. Kirillov

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1. Definitions and notations.

The group S_n, $n \geq 1$, is usually defined as the group of all **permutations** of the set $X_n = \{1, 2, \dots, n\}$. So, an element $s \in S_n$ is a bijection (i.e., one-to-one map) from X_n to $X_n : k \mapsto s(k)$. The composition of two such map is denoted by $s_1 \circ s_2 : k \mapsto s_1\big(s_2(k)\big)$.

It is convenient to depict the permutation s as follows. Take two copy of X_n represented as column vectors and draw a system of n arrows joining element k of the first copy with the element $s(k)$ of the second copy.

To depict the composition $s_1 \circ s_2$ we use three copies of X_n. Then we join elements of the first copy with elements of the second one according to map s_1 and elements of the second copy with elements of the third one according to map s_2. Finally, we erase the second copy and "straighten" the broken lines joining elements of the first and third copies. We get the picture of $s_1 \circ s_2$.

At this place it is convenient to define an important characteristic of a permutation s. Namely, the number of intersection points of all arrows $k \mapsto s(k)$, $1 \leq k \leq n$, is called the **length** of s and is denoted by $l(s)$.

Exercise 1. a) Describe all permutation of length 0; b) Describe all permutation of length 1; c) What is the maximal length of an $s \in S_n$?

Denote by σ_i, $1 \leq i \leq n-1$, the permutation which exchanges i and $i+1$ and fixes all other elements of X_n.

Exercise 2. Show that elements σ_i, $1 \leq i \leq n-1$, generate the group S_n and satisfy the relations

$$(1) \qquad \sigma_i^2 = e; \qquad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}; \qquad \sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{for} \quad |i-j| \geq 2.$$

Theorem 1. An element $s \in S_n$ has length $l(s) \leq k$ if and only if it can be written as a product of $\leq k$ generators $\sigma_1, \sigma_2, \dots, \sigma_n$.

Proof. From the definition of $s_1 \circ s_2$ we derive that

$$l(s_1 \circ s_2) = l(s_1) + l(s_2) - 2m,\ m \geq 0. \tag{2}$$

Indeed, if the arrows $i \mapsto s_1 \circ s_2(i)$ and $j \mapsto s_1 \circ s_2(j)$ have no intersections, then sign of $i - j$ equal to the sign of $s_1 \circ s_2(i) - s_1 \circ s_2(j)$. Then the broken lines $i \mapsto s_1(i) \mapsto s_1 \circ s_2(i)$ and $j \mapsto s_1(j) \mapsto s_1 \circ s_2(j)$ have zero or 2 intersections depending on the sign of $s_1(i) - s_1(j)$ (draw a picture).

If the arrows $i \mapsto s_1 \circ s_2(i)$ and $j \mapsto s_1 \circ s_2(j)$ have one intersection, then the broken lines $i \mapsto s_1(i) \mapsto s_1 \circ s_2(i)$ and $j \mapsto s_1(j) \mapsto s_1 \circ s_2(j)$ have also 1 intersection (prove it yourself, considering two possible signs of $s_1(i) - s_1(j)$).

Now, assume that $s = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}$. Then $l(s) \leq k$ because of (2). Conversely, if $l(s) \leq k$, consider the first intersection point (going from left to right). Let its height is between i and $i+1$, then $s' = \sigma_i \circ s$ has the property $l(s') = l(s) - 1$. So s' is a product of $\leq k-1$ generators, hence, $s = \sigma_i \circ s'$ is a product of $\leq k$ generators.
□

Another corollary from (2) is that the map

$$\mathrm{sgn}:\ S_n \to \{\pm 1\}; \quad \mathrm{sgn}\,(s) = (-1)^{l(s)} \tag{3}$$

is multiplicative:

$$\mathrm{sgn}\,(s_1 \circ s_2) = \mathrm{sgn}\,(s_1) \cdot \mathrm{sgn}\,(s_2). \tag{4}$$

A permutation s is called **even** (resp. **odd**) if $\mathrm{sgn}\, s = 1$ (resp. $\mathrm{sgn}\, s = -1$). The set of even permutations form a normal subgroup $A_n \subset S_n$ of index 2.

2. Lagrange theorem and applications.

We say that a group G **acts from the left** on a set X if to each $g \in G$ there corresponds a transformation $T(g) : x \mapsto g \cdot x$, so that $T(g_1 g_2) = T(g_1)T(g_2)$, or $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$. In this case X is called a **left G-space**.

The **right G-space** is defined analogously, but now for $g \in G$ we associate the transformation $T'(g) : x \mapsto x \cdot g$ so that $T'(g_1 g_2) = T'(g_2)T'(g_1)$.

Any left action can be transformed into right action via the following rule.

Exercise 3. Show that if $g \mapsto T(g)$ is a left action, then $g \mapsto T(g^{-1}) = T^{-1}(g)$ is a right action.

We say that G acts on X **transitively**, or, that X is an **homogeneous** G-space, if for any two points $x_1,\ x_2 \in X$ there is a $g \in G$ such that $T(g)x_1 = x_2$.

Suppose now that X is an homogeneous G-space with a marked point $x_0 \in X$. Let

$$\mathrm{Stab}(x_0) := \{g \in G \mid T(g)x_0 = x_0\}. \tag{5}$$

Then $\mathrm{Stab}(x_0)$ is a subgroup of G. Conversely, for any subgroup $H \subset G$ there exists an homogeneous space X with a marked point $x_0 \in X$ such that $\mathrm{Stab}(x_0) = H$.

To construct X explicitly, we assume that it exists and consider the set $G(x)$ of all elements $g \in G$ which send x_0 to x. For definiteness we suppose that X is a left G-space. Let $g(x)$ be a representative of the set $G(x)$. Take any element $g \in G(x)$ and compare it with $g(x)$. Since $g \cdot x_0 = x = g(x) \cdot x_0$, we get $\big(g(x)^{-1}g\big)x_0 = x_0$, hence, $g(x)^{-1}g \in H$. Thus, $G(x) = gH = \{gh \mid h \in H\}$. The subsets of the form gH are called **left H-cosets** in G. We see, that points of X are in a bijection with left H-cosets in G.

Now we can define our homogeneous space X as a collection of all left H-cosets in G. Usually this collection is denoted by G/H. There is a natural left action of G on G/H : $g_1 \cdot (g_2H) = (g_1g_2)H$. The role of a marked point is played by the H-coset $H \subset G$.

In the case when G is a finite group, we have the following equality

$$|G| = |H| \cdot |G/H| \tag{6}$$

where $|X|$ denote the number of points in a finite set X.

Corollary (Lagrange Theorem). The order of a subgroup is a divisor of the order of a group.

Exercise 4. Compute orders of the groups:
S_n, A_n, $Rot(P)$, $Iso(P)$
where P is a regular polytope in $\mathbb{R}^3$, Rot is the group of rotation and Iso is the group of all isometries (rotations and reflections).

3. Conjugacy classes of S_n.

A map $\varphi : G \to G$ is called an **automorphism** if it is a bijection and preserves the multiplication law: $\varphi(g_1g_2) = \varphi(g_1)\varphi(g_2)$. The collection of all automorphisms form a group denoted by $\mathrm{Aut}(G)$.

For any $x \in G$ the map $\varphi_x(g) := xgx^{-1}$ is an automorphism of G. Such automorphisms are called **inner** and form a subgroup $\mathrm{Inn}(G) \subset \mathrm{Aut}(G)$.

Exercise 5. Show that $\mathrm{Inn}(G)$ is a normal subgroup in $\mathrm{Aut}(G)$.

The quotient group $\mathrm{Aut}(G)/\mathrm{Inn}(G)$ is denoted $\mathrm{Out}(G)$ and its elements are called **outer** automorphisms. If G is abelian, all automorphisms are outer. For some groups, e.g. for all S_n, $n \neq 6$, all automorphisms are inner.

We say that two elements g_1, $g_2 \in G$ are **conjugate** and write $g_1 \sim g_2$ if there exists an $x \in G$ such that $\varphi_x(g_1) = g_2$. In this case $g_2 = x^{-1}g_1x$ and, denoting $x^{-1}g_1$ by y, we can write $g_1 = xy$, $g_2 = yx$. Conversely, for any x, $y \in G$ we have $xy \sim yx$.

A non-formal meaning of the conjugacy relation: two transformations are conjugate in G if they look the same for two observers, one of which is obtained from other by a transformation from G.

Thus, any group G splits into conjugacy classes $C_0 = \{e\}$, C_1, C_2, $\ldots$, C_k. The set of conjugacy classes we denote by $Cl(G)$.

Each conjugacy class $C \subset G$ is an homogeneous G-space where G acts by inner automorphisms. The stabilizer of a point $g \in C$ is the **centralizer** of g in G, denoted by $Z_G(g)$ and defined by:

$$Z_G(g) = \{x \in G \mid xg = gx\}. \tag{7}$$

So, the cardinality of a conjugacy class $C \subset G$ is always a divisor of the order of the group G.

To go further we need a notion of a partition. Consider a sequence

$$\lambda = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0\} \quad \text{with} \quad \sum_{k=1}^{r} \lambda_k = n.$$

Such a sequence is called a **partition** of n. Usually a partition of n is depicted by a **Young diagram**[1] with n boxes, arranged in r rows so that k^{th} row contains λ_r boxes.

If we count boxes not by rows but by columns, we get another partition of n, $\mu = \{\mu_1 \geq \mu_2 \geq \cdots \geq \mu_s > 0\}$ which is called **conjugate** or **dual** to λ. The explicit formula is

$$\mu_k = \operatorname{Card}\{\lambda_i \mid \lambda_i \geq k\}. \tag{8}$$

Finally, we can count the number α_k of cycles of size k and denote partition λ by $1^{\alpha_1} 2^{\alpha_2} \cdots n^{\alpha_n}$. Here $\alpha_k = \mu_k - \mu_{k+1}$.

Theorem 2. Let μ be a partition of n. Denote by c_μ the permutation

$$c_\mu = c_{\mu_1} c_{\mu_2} \cdots c_{\mu_k} \tag{9}$$

where $c_{\mu_i} \in S_n$ permutes cyclically numbers from $m_1 + m_2 + \cdots + m_{i-1} + 1$ to $m_1 + m_2 + \cdots + m_i$ and leave fixed all other numbers in X_n.

Any permutation $s \in S_n$ is conjugate to a unique c_μ, $|\mu| = n$.

Proof. Take any permutation $s \in S_n$ and consider the subgroup $\langle g \rangle$ generated by g. It is a cyclic group of some order k, which is a divisor of $n = |G|$. But any subgroup of the cyclic group is itself a cyclic group. Hence, orbits of $\langle g \rangle$ in X_n are all of the form $\Omega_i = \langle g \rangle / \langle g^{m_i} \rangle$ where m_i is a divisor of k. We can assume that $m_1 \geq m_2 \geq \cdots \geq m_k$. This partition of n is called the **cycle structure** of g.

Now we observe that an inner automorphism of S_n can be interpreted as a renaming of elements of X_n. We can denote the elements of Ω_1 by numbers 1, 2, …, m_1, the elements of Ω_2 by numbers $m_1 + 1$, $m_1 + 2$, …, $m_1 + m_2$ and so on. It follows that up to inner automorphism, the element g is determined by its cycle structure. Therefore, we have a natural labeling of $Cl(G)$ by partitions of n.

□

We denote by C_μ the cojugacy class which contains c_μ.

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[1] The name **Ferrer diagram** is historically more correct but Young diagram is more popular.

4. Some computations.

Let us compute the specific cardinality of a conjugacy class $C^\alpha \subset S_n$. For this, because of Lagrange theorem, we have to now the cardinality of $Z_{S_n}(s)$ for $s \in C^\alpha$ If $x \in Z_{S_n}(s)$, then it can only permute the cycles of equal length and make a shift in each cycle. Therefore, the total number of such elements is

$$\operatorname{Card} Z_{S_n}(s) = \prod_{k=1}^{n} (\alpha_k)! k^{\alpha_k}$$

and the specific cardinality of the conjugacy class is

$$\frac{\operatorname{Card} C^\alpha}{\operatorname{Card} S_n} = \frac{1}{\operatorname{Card} Z_{S_n}(s)} = \frac{1}{\prod_{k=1}^{n} (\alpha_k)! k^{\alpha_k}}. \tag{10}$$

We want to introduce a generating function for specific cardinalities of conjugacy classes in S_n:

$$C_n(t_1, t_2, \ldots, t_n) := \sum_{\sum_{k=1}^{n} k\alpha_k = n} \frac{\operatorname{Card} C^\alpha}{\operatorname{Card} S_n} \cdot t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_n^{\alpha_n}. \tag{11}$$

It is more convenient to drop the restriction $\sum_{k=1}^{n} k\alpha_k = n$. For this we can consider a more universal generating function

$$\mathcal{C}(t_1, t_2, \ldots, t_n; \lambda) := \sum_{n \geq 0} \lambda^n \cdot C_n(t_1, t_2, \ldots, t_n). \tag{12}$$

To compute it, we multiply (10) by $\prod_{n\geq 0} (\lambda t_k)^{k\alpha_k}$ and sum up over all $\alpha_k \geq 0$ without restrictions. The result is

$$\sum_{\alpha_k \geq 0} \prod_{k \geq 1} \frac{(\lambda^k t_k)^{\alpha_k}}{(\alpha_k)! k^{\alpha_k}} = \prod_{k \geq 1} \sum_{\alpha_k \geq 0} \frac{(\lambda^k t_k)^{\alpha_k}}{(\alpha_k)! k^{\alpha_k}} = \prod_{k \geq 1} \exp \frac{\lambda^k t_k}{k} = \exp \sum_{k \geq 1} \frac{\lambda^k t_k}{k}. \tag{13}$$

In particular, for $t_1 = t_2 = \cdots = 1$ we obtain

$$\mathcal{C}(1, 1, \ldots; \lambda) = \exp \sum_{k \geq 1} \frac{\lambda^k}{k} = (1-\lambda)^{-1} = 1 + \lambda + \lambda^2 + \ldots.$$

We see that the coefficient by any power of λ is 1 which shows that sum of specific cardinalities of conjugacy classes is 1, as it must be.

The same type of computation allows to solve explicitly many interesting problems about the structure of S_n.

We give an example: what is the number Inv (n) of involutions in the group S_n? It is clear that involutions are characterized by the condition $\alpha_3 = \alpha_4 = \ldots = 0$. So, repeating the summation under this restriction, we get

$$\sum_{n \geq 0} \frac{\operatorname{Inv}(n)}{n!} \lambda^n = \exp \sum_{k=1,2} \frac{\lambda^k}{k} = e^{\lambda + \frac{\lambda^2}{2}}. \tag{14}$$

Thus, Inv (n) is equal to $n! \times$ coefficient by λ^n in $e^{t+\frac{t^2}{2}}$, i.e.

$$\sum_{l=0}^{[\frac{n}{2}]} \frac{n!}{l!(n-2l)!2^l} = 1 + \frac{n(n-1)}{1\cdot 2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4} + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{6\cdot 8} + \dots. \tag{15}$$

Exercise 6. Find the number of permutations in S_{2n} which contain only circles of even length.

Answer: $\big((2n-1)!!\big)^2$.

5. Intertwining operators.

Let (π_1, V_1) and (π_2, V_2) be two linear representations of a group G. An operator $A : V_1 \to V_2$ is called **intertwining operator**, or simply **intertwiner**. if the following diagram is commutative:

$$\begin{tikzcd} V_1 \arrow[r, "A"'] \arrow[d, "\pi_1(g)"'] & V_2 \arrow[d, "\pi_2(g)"] \\ V_1 \arrow[r, "A"'] & V_2 \end{tikzcd}$$

The set of all intertwiners for π_1, π_2 forms a complex vector space $I(\pi_1, \pi_2)$. Its dimension is denoted by $i(\pi_1, \pi_2)$ and is called **intertwining number**.

Note, that for unitary complex representations the hermitian conjugation sends $A \in I(\pi_1, \pi_2)$ to $A^* \in I(\pi_2, \pi_1)$. So, $i(\pi_1, \pi_2)$ is symmetric.

Theorem (Schur Lemma) For two irreducible representations we have

$$i(\pi_1, \pi_2) = \begin{cases} 1 & \text{if} \quad \pi_1 \sim \pi_2 \\ 0 & \text{otherwise} \end{cases}$$

We can consider intertwining number as a sort of inner product for representations for which the irreducible representations play the role of elements of an orthonormal basis. Indeed, if $\{\pi_1, \pi_2, \dots, \pi_k\}$ are all (up to equivalence) irreducible representations of a finite group G, then any representation π is equivalent to the direct sum of irreducible components where π_j enters with multiplicity $m_j = i(\pi, \pi_j)$. We write it in the form $\pi = \sum_{j=1}^k m_j \pi_j$. If $\pi' = \sum_{j=1}^k m'_j \pi_j$ is any other representation, then $i(\pi, \pi') = \sum_{j=1}^k m_j m'_j$. In particular, $i(\pi, \pi) = 1$ iff π is irreducible.

The analogy between intertwining number and inner product becomes equality if we pass from representations to their characters. Namely, let $\chi_j = \operatorname{tr} \pi_j$ be the character of π_j. Then in $L_2(G)$, the space of complex-valued functions on G with inner product $(f_1, f_2) = \frac{1}{|G|} \sum_{g\in G} f_1(g)\overline{f_2(g)}$, we have $(\chi_j, \chi_{j'}) = \delta_{jj'}$. It follows from orthogonality relations between matrix elements of irreducible representations, which we supposed to be known.

6. Intertwining operators for geometric representations.

A **geometric representation** of a finite group G is related to any G-space X. If X is a right G-space, it acts in the space V_X of all complex-valued functions on X by the formula

$$\big(\pi_X(g)f\big)(x) = f(x \cdot g). \tag{16}$$

For a left G-space the formula is

$$\big(\pi_X(g)f\big)(x) = f(g^{-1} \cdot x). \tag{16'}$$

Usually we use right G-space to avoid the inverse operation in (16′).

Sometimes, geometric representations are called **permutation representations** because in the natural basis in V_X (see below) the operators $\pi_X(g)$ just permute basic vectors.

Let now $X = G/H$ and $Y = G/K$ be two right G-spaces. Consider geometric representations (π_X, V_X) and (π_Y, V_Y) and an intertwining operator $A: V_X \to V_Y$. It can be written in the form

$$(Af)(y) = \sum_{x \in X} a(x, y) f(x) \tag{17}$$

where a is a complex-valued function on $X \times Y$. It can be explicitly written as

$$a(x, y) = (A\delta_x)(y), \quad \text{where} \quad \delta_x(x') = \begin{cases} 1 & \text{if} \quad x' = x, \\ 0 & \text{if} \quad x' \neq x. \end{cases}$$

The condition $A\pi_X(g) = \pi_Y(g)A$ looks like

$$\sum_{x \in X} a(x, y) f(x \cdot g) = \sum_{x \in X} a(x, y \cdot g) f(x) \overset{x \mapsto x \cdot g}{=} \sum_{x \in X} a(x \cdot g, y \cdot g) f(x \cdot g),$$

or

$$a(x, y) = a(x \cdot g, y \cdot g) \quad \text{for all} \quad x \in X,\, y \in Y,\, g \in G. \tag{18}$$

So, the function a is constant on G-orbits in $X \times Y$.

Assume now that X and Y are right homogeneous spaces, so that $X \sim H\backslash G$, and $Y \sim K\backslash G$.

Lemma 1. The following four sets are naturally isomorphic:

a) the set Y/H of H-orbits in Y;

b) the set X/K of K-orbits in X;

c) the set $(X \times Y)/G$ of G-orbits in $X \times Y$.

d) the set $H\backslash G/K$ of double (H, K)-cosets in G, i.e. subsets of the form HgK.

Proof. For any $g \in G$ we define the following four objects:
a) H-orbit $\Omega'(g) \in Y/H$ which contains the element $Kg^{-1} \in Y$;
b) K-orbit $\Omega''(g) \in X/K$ which contains the element $Hg \in X$;
c) G-orbit $\Omega(g) \in (X \times Y)/G$ which contains the element $(Hg,\, K) \in X \times Y$;
d) the double class HgK.

It remains to check that $\Omega(g)$, $\Omega'(g)$, $\Omega''(g)$ actually depend only on the double class HgK (i.e. do not change if we shift g by an element $h \in H$ from the left or by an element $k \in K$ from the right). We leave it to the reader.

It follows that

$$i(\pi_X,\, \pi_Y) = \mathrm{Card}\,(H\backslash G/K). \tag{19}$$

In particular, this number is always ≥ 1.

7. Young subgroups in S_n.

For any partition λ of n we define the abstract group Y_λ as the product

$$Y_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_r}.$$

Let P_λ be the set of all partitions p of X_n onto r parts $p_1, p_2, \ldots, p_r$ of size $\lambda_1, \lambda_2, \ldots, \lambda_r$. The group S_n acts naturally on P_λ and for the convenience of notations we choose the right action. So, a partition p goes to the partition $p \cdot s$. If a part p_i consists of numbers $k_1, \ldots, k_{\lambda_i}$, then the part $(p \cdot s)_i$ consists of $s^{-1}(k_1), \ldots, s^{-1}(k_{\lambda_i})$.

It must be clear (think it through!) that P_λ is an homogeneous space and the stabilizer $\mathrm{Stab}\, p$ of any point $p \in P_\lambda$ is isomorphic to Y_λ.

Let V_λ be the space of all complex-valued functions on P_λ. We define two linear representations $\pi_\lambda^\pm$ of the group S_n in V_λ by the formulas

$$\big(\pi_\lambda^+(s)f\big)(p) = f(p \cdot s) \qquad \big(\pi_\lambda^-(s)f\big)(p) = \mathrm{sgn}\,(s)f(p \cdot s). \tag{20}$$

So, π_λ^+ is a geometric representation associated with the homogeneous G-space P_λ and $\pi_\lambda^- = \pi_\lambda^+ \otimes \mathrm{sgn}$.

The dimension of $\pi_\lambda^\pm$ is

$$|P_\lambda| = \frac{|S_n|}{|Y_\lambda|} = \frac{n!}{\prod_{k=1}^r (\lambda_k)!} = \frac{n!}{\prod_{k=1}^n (k!)^{\alpha_k}}.$$

To visualize elements of P_λ we can consider a Young diagram of shape λ and fill it up by numbers from 1 to n. The resulting object is called a **Young tableau** t. Let us call two tableaux t_1, t_2 **row equivalent** if one of them can be obtained from the other by permutation of numbers inside rows. An equivalence class of tableaux is called a **tabloid** and is denoted by bold letter $\mathbf{t}$. To write a tabloid, we erase the boundaries between boxes situated in the same raw in t. It gives us a partition $p_{\mathbf{t}}$ of X_n into parts consisting of numbers situated in the same row. Since the relation $\mathbf{t} \longleftrightarrow p_{\mathbf{t}}$ between tabloids and partitions of the same shape is one-to-one, we shall identify them, so that the set P_la is the same as the set T_λ of all tabloids of the shape λ.

8. Intertwining operators for π_λ^+ and π_μ^-.

Come back to the group S_n and apply the technique above to intertwiner A between a geometric representation π_λ^+ and a "twisted" geometric representation π_μ^-. It corresponds to a function a on $P_\lambda \times P_\mu$ which satisfies the "twisted" condition (18):

$$(18') \qquad a(p\cdot s,\, q\cdot s) = \operatorname{sgn}(s) a(p,\, q) \quad \text{for all} \quad p \in P_\lambda, \quad q \in P_\mu, \quad g \in G.$$

Let us realize points $p \in P_\lambda$, filling a Young diagram of shape λ by numbers $1,\, 2,\, \ldots,\, n$ so that numbers increase in each row from left to right. The points $g \in P_\mu$ we realize, filling a Young diagram μ', which is conjugate to μ, so that numbers increase in each column from top to bottom.

To go further we need a two new notions: a full and a partial orders on the set of all partitions of n. The full order is called **lexicographical** and is used in dictionaries. Namely, we say that a partition λ is bigger than μ and write $\lambda > \mu$, if one of the following is true:

$$(21) \qquad \begin{aligned} &\lambda_1 > \mu_1; \\ &\lambda_1 = \mu_1,\, \lambda_2 > \mu_2; \\ &\lambda_1 = \mu_1,\, \lambda_2 = \mu_2,\, \lambda_3 > \mu_3; \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\lambda_1 = \mu_1,\, \lambda_2 = \mu_2,\, \ldots,\, \lambda_{r-1} > \mu_{r-1}. \end{aligned}$$

A partial order is called **dominance**. Namely, we say that a partition λ **dominates** μ and write $\lambda \succ \mu$ if $\lambda \neq \mu$ and for any $k \geq 1$ we have

$$(22) \qquad \lambda_1 + \lambda_2 + \cdots + \lambda_k \geq \mu_1 + \mu_2 + \cdots + \mu_k.$$

We use also the notations $\lambda \geq \mu$ and $\lambda \succeq \mu$ in the obvious sense.

Exercise 7. Show that $\lambda \succ \mu$ implies $\lambda > \mu$, but the converse is not always true.

Main Theorem. The intertwining number $i_{\lambda,\mu} := i(\pi_\lambda^+,\, \pi_\mu^-)$ has the properties:

a) $i_{\lambda,\mu} = 0 \quad \text{if} \quad \lambda' \not\succeq \mu$ (in particular, if $\lambda < \mu$);

b) $i_{\lambda,\lambda'} = 1 \quad \text{for all} \quad \lambda$.

Proof. Consider an intertwiner $A \in I(\pi_\lambda^+,\, \pi_\mu^-)$. It is given by a function $a(p,\, q),\ p \in P_\lambda,\ q \in P_\mu$, such that

$$(23) \qquad a(p\cdot s,\, q\cdot s) = \operatorname{sgn}(s) a(p,\, q).$$

Assume that $a(p_0,\, q_0) \neq 0$. Then if two different numbers k, l occurs in the same element of partition q_0, they must belong to the different elements of partition p_0. Otherwise, the transposition $(k\,l)$ belongs simultaneously to stabilizers of p_0 and of q_0 and from (21) we obtain $a(p_0,\, q_0) = \operatorname{sgn}(k\,l)\, a(p_0,\, q_0) = -a(p_0,\, q_0)$. A contradiction.

It follows that number of parts in p_0 can not be less than the maximal part of q_0. In other words we have $\lambda_1' \geq \mu_1$.

Further, the elements of the second part of q_0 also must belong to different part of p_0. Therefore, the $\mu_1 + \mu_2$ numbers from two biggest parts of μ are distributed between parts of p_0 so, that no part gets more than two elements. It means that $\lambda'_1 + \lambda'_2 \geq \mu_1 + \mu_2$.

Continue this argument, we see that the necessary condition for $a(p_0,\, q_0) \neq 0$ is $\lambda' \succeq \mu$. I leave you to verify that it is also a sufficient condition.

To consider now the case $\lambda' = \mu$, we introduce some more terminology.

If we fill a Young diagram D_λ of shape λ by numbers from X_n, we get a **tableau** T of shape λ. Two tableaux are called **row equivalent** if one can be obtained from another by a permutations in every row.

Any partition p of X_n of shape λ is a partition into the rows of some tableau T of shape λ. The tableau T is defined by p up to row equivalence. Since $\lambda = \mu'$, the partition q is a partition into rows of some tableau T' of shape $\mu = \lambda'$.

We know that $a(p,\, q) \neq 0$, if only all elements of the first row in T' belong to different rows in T. Using the row equivalence, we can replace T by some tableau T_1 so that elements of the first row in T' occupy exactly the first column of T_1.

Now, elements of the second row of T' are also belong to different rows of T and of T_1. Passing from t_1 to a row equivalent tableau T_2, we can assume that these elements belong to the second column of T_2.

Continuing this procedure, we come to a tableau $\widetilde{T} := T_{\lambda_1}$ such that p is the partition of X_n into the rows of $\widetilde{T}$ and q is a partition of X_n into the columns of $\widetilde{T}$.

Note, that the action of $s \in S_n$ on the pair $(p,\, q) \in P_\lambda \times P_{\lambda'}$ sends the tableau $\widetilde{T}$ to another tableau $\widetilde{T} \cdot s$ which is obtained from $\widetilde{T}$ by replacing k by $s^{-1}(k)$, $1 \leq k \leq n$. We see that the function $a(p,\, q)$ is different from zero only on one S_n-orbit in $P_\lambda \times P_{\lambda'}$. It proves the second statement of the theorem.

□

9. Big subgroups.

We call a subgroup $H \subset G$ **big subgroup** if any unirrep $(\pi,\, V)$ of G being restricted to H has a simple spectrum, i.e. splits into non-equivalent unirreps of H.

Theorem 3. For any $n \geq 1$ the group S_n is a big subgroup in S_{n+1}.

Lemma 2. Let G be a finite group and let $\pi_1,\, \pi_2,\, \ldots,\, \pi_k$ be the whole list of unirreps of G up to equivalence. Assume that a unirep π of G has a decomposition

$$\pi = m_1 \cdot \pi_1 + m_2 \cdot \pi_2 + \ldots + m_k \cdot \pi_k. \tag{24}$$

Then the algebra $I(\pi,\, \pi)$ of intertwining operators is isomorphic to the algebra

$$\mathrm{Mat}_{m_1}(\mathbb{C}) \oplus \mathrm{Mat}_{m_2}(\mathbb{C}) \oplus \ldots \oplus \mathrm{Mat}_{m_k}(\mathbb{C}). \tag{25}$$

Proof. In an appropriate basis the matrix of $\pi(g)$ have a bloc-diagonal form with $m_i \times m_i$ blocks like

$$\begin{pmatrix} \pi_i(g) & 0 & 0 & \ldots & 0 & 0 \\ 0 & \pi_i(g) & 0 & \ldots & 0 & 0 \\ 0 & 0 & \pi_i(g) & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & 0 & \pi_i(g) \end{pmatrix}.$$

Let A be an intertwiner, written in the same basis. From Schur lemma we know that $I(\pi_i, \pi_j) = \begin{cases} \mathbb{C} & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j. \end{cases}$

It follows that A is also a block-diagonal matrix whose i-th block has the form

$$\begin{pmatrix} a^i_{1,1} \cdot 1 & a^i_{1,2} \cdot 1 & \dots & a^i_{1,m_i} \cdot 1 \\ a^i_{2,1} \cdot 1 & a^i_{2,2} \cdot 1 & \dots & a^i_{2,m_i} \cdot 1 \\ \dots & \dots & \dots & \dots \\ a^i_{m_i,1} \cdot 1 & a^i_{m_i,2} \cdot 1 & \dots & a^i_{m_i,m_i} \cdot 1 \end{pmatrix}$$

where a^i_{jk} are arbitrary complex numbers. So, the intertwiners, corresponding to this block form an algebra isomorphic to $\mathrm{Mat}_{m_i}(\mathbb{C})$.

$\square$

Corollary. The algebra $I(\pi, \pi)$ is commutative if and only if π has a simple spectrum (i.e. all multiplicities m_i are ≤ 1).

Let G be a finite group. Recall that the group algebra $\mathbb{C}[G]$ consists of all complex-valued functions on G with the ordinary structure of a complex vector space and with a non-standard multiplication, denoted by $*$ and called **convolution**. By definition,

$$(f_1 * f_2)(g) = \sum_{h \in G} f_1(h) f_2(h^{-1} g). \tag{26}$$

Let us denote by δ_g a function on G given by $\delta_g(h) = \begin{cases} 1 & \text{for} \quad h = g \\ 0 & \text{otherwise.} \end{cases}$

Exercise 8. Show that $\delta_{g_1} * \delta_{g_2} = \delta_{g_1 g_2}$.

We define an operation $^\vee$ on $\mathbb{C}[G]$ by $f^\vee(g) := f(g^{-1})$.

Exercise 9. Show that $(f_1 * f_2)^\vee = f_2^\vee * f_1^\vee$ so that $*$ is an **antiinvolution**.

We denote by $\mathcal{A}(G)$ the subspace in $\mathbb{C}[G]$ consisting of functions satisfying the equation $f = f^\vee$. From Exercise 9 we derive

Lemma 3. Any subalgebra of $\mathbb{C}[G]$ which is contained in $\mathcal{A}(G)$ is commutative.

$\square$

Denote by $\mathbb{C}[S_{n+1}]^{S_n}$ the centralizer of S_n in $\mathbb{C}[S_{n+1}]$, i.e. collection of functions satisfying

$$f(x^{-1} g x) = f(s) \quad \text{for all} \quad s \in S_{n+1},\, x \in S_n. \tag{27}$$

Lemma 4. The algebra $\mathbb{C}[S_{n+1}]^{S_n}$ is contained in $\mathcal{A}(S_{n+1})$, hence, by Lemma 3, is commutative.

Proof. We derive the property $f = f^\vee$ from (27). For this end we consider a given element $s \in S_{n+1}$ as a product of cycles. We want to find an element $x \in S_n$ such that $x^{-1} s x = s^{-1}$. Such an element must reverse the order for each cycle and fix the number $n + 1 \in X_{n+1}$. Let us draw all cycles of s as regular polygons with centers on one line L; moreover, we can arrange that the number $n + 1$ is also on the same lane. Then the reflection in L gives the desired element x.

Exercise 10. a) Prove that the algebra $\mathbb{C}[S_{n+2}]^{S_n \times S_2}$ is commutative for all n.
b) Check that the algebra $\mathbb{C}[S_6]^{S_3 \times S_3}$ is not commutative.

We return to the proof of the theorem. We want to show that for any unirrep (π, V) of S_{n+1} the restriction $\pi' := \mathrm{Res}_{S_n}^{S_{n+1}}\pi$ has a simple spectrum, or, according to the Lemma 1, that the algebra $I(\pi', \pi')$ is commutative.

First, we extend the representation (π, V) from S_{n+1} to the group algebra $\mathbb{C}[S_{n+1}]$. Then we use the Wedderburn theorem which claims that the only irreducible subalgebra in $\mathrm{End}\, V \simeq \mathrm{Mat}_n(\mathbb{C})$ is the whole algebra. The conclusion is that any linear operator $A \in \mathrm{End}\, V$ has the form $\pi(f)$ for some function $f \in \mathbb{C}[S_{n+1}]$.

Lemma 5. If $A \in I(\pi', \pi')$, then the function f above can be chosen from $\mathbb{C}[S_{n+1}]^{S_n}$.

Indeed, consider the operator P which sends $f \in \mathbb{C}[S_{n+1}]$ to the function

$$(Pf)(s) = \frac{1}{|S_n|} \sum_{x \in S_n} f(x^{-1}sx).$$

It is easy to check that P is a projection of $\mathbb{C}[S_{n+1}]$ onto $\mathbb{C}[S_{n+1}]^{S_n}$. On the other hand, for an intertwiner $A = \pi(f)$ we have

$$\pi(Pf) = \frac{1}{|S_n|} \sum_{x \in S_n} \pi(x)\pi(f)\pi(x^{-1}) = \frac{1}{|S_n|} \sum_{x \in S_n} \pi(x)A\pi(x^{-1}) = A$$

and we can replace f by Pf.

Now, we can finish the proof of the theorem. Indeed the algebra $I(\pi', \pi')$ is a homomorphic image of the commutative algebra $\mathbb{C}[S_{n+1}]^{S_n}$, hence, is commutative.

□

10. Structure of $\widehat{S_n}$.

From the Main Theorem of section 8 we derive now the classification and some properties of **unirreps**[2] of S_n. First, we have that the set $\widehat{S_n}$ of equivalence classes of all unirreps is naturally labelled by partitions of n. Indeed, let π_λ denote the class of the only common irreducible component for π_λ^+ and $\pi_{\lambda'}^-$.

Theorem 4. $\widehat{S_n} = \{\pi_\lambda,\, |\lambda| = n\}$.

First, we show that π_λ are pairwise distinct. Suppose the contrary: $\pi_\lambda \sim \pi_\mu$ and $\lambda \neq \mu$. Then $\lambda' \neq \mu'$ and we can assume that $\lambda' < \mu'$. On the other hand, the representations π_λ^+ and π_μ^- have common component $\pi_\lambda \sim \pi_\mu$. It can be only if $\mu' \prec \lambda'$, hence, $\mu' < \lambda'$. A contradiction.

It remains to observe that $|\widehat{S_n}|$, the number of non-equivalent unirreps, is equal to $|Cl(S_n)|$, the number of conjugacy classes. But we have seen that the latter is equal to the number $p(n)$ of partitions of n. So, the representations π_λ exhaust $|\widehat{S_n}|$.

□

[2] This abbreviation means "unitary irreducible representation".

More detailed analysis of the proof of the Main Theorem in section 8 shows that the partition λ can be determined by π_λ in two ways:
a) λ is the maximal element of the set $\{\mu \mid |\mu| = n \quad \& \quad i(\pi^+_\mu, \pi_\lambda) \neq 0\}$;
b) λ' is the minimal element of the set $\{\mu \mid |\mu| = n \quad \& \quad i(\pi^-_\mu, \pi_\lambda) \neq 0\}$.

The case $n = 4$ is illustrated below.

Table of intertwining numbers

(28)

$\lambda\backslash\mu$	π^+_4	π^+_{31}	π^+_{22}	$\pi^+_{21^2}$	$\pi^+_{1^4}$	π^-_4	π^-_{31}	π^-_{22}	$\pi^-_{21^2}$	$\pi^-_{1^4}$
π_4	1	1	1	1	1	1	0	0	0	0
π_{31}	3	2	1	1	0	3	1	0	0	0
π_{22}	2	1	1	0	0	2	1	1	0	0
π_{21^2}	3	1	0	0	0	3	2	1	1	0
π_{1^4}	1	0	0	0	0	1	1	1	1	1

Note that π^+_λ, being geometric representations, have integer-valued characters. The characters of representations π^-_λ are obtained by multiplication by sgn, hence, also take integer values. On the other hand, representations π^+_λ and π_λ are related by unitriangular matrix of intertwining numbers whose inverse is also unitriangular. We obtain

Theorem 5. All representations of S_n have integer-valued characters.

Moreover, any representation in an appropriate basis can be written by matrices with integer entries.

11. Representation of S_n in the space $\mathbb{C}[x_1, x_2, \ldots, x_n]$.

Another source of interesting representations of S_n is the polynomial algebra $\mathbb{C}[x_1, \ldots, x_n]$. It is infinite dimensional space but admits a $\mathbb{Z}_+$-grading with finite dimensional homogeneous components.

Exercise 11. Show that $\dim \mathbb{C}^k[x_1, \ldots, x_n] = \binom{n+k-1}{k} = (-1)^k\binom{-1}{k}$.

The analysis of intertwining numbers $i(V_\lambda, \mathbb{C}^k[x_1, \ldots, x_n])$ and their generating functions

$$P_\lambda(t) := \sum_{k\geq 0} i(V_\lambda, \mathbb{C}^k[x_1, \ldots, x_n])t^k$$

is a very deep and beautiful problem. For initial values $n = 1, 2, 3$ we get

(29)
$$P_1(t) = \frac{1}{1-t};$$
$$P_2(t) = \frac{1}{(1-t)(1-t^2)}, \qquad P_{1,1}(t) = \frac{t}{(1-t)(1-t^2)};$$
$$P_3 = \frac{1}{(1-t)(1-t^2)(1-t^3)}, \quad P_{2,1}(t) = \frac{t}{(1-t)^2(1-t^3)}, \quad P_{1,1,1}(t) = \frac{t^3}{(1-t)(1-t^2)(1-t^3)}.$$

To formulate the general fact, we introduce some notation. Consider a box $\Box$ in a Young diagram D_λ of shape λ. We numerate the boxes by pairs of non-negative integers $\big(i(\Box),\, j(\Box)\big)$ where $i(\Box)$ denotes the number of the row, and $j(\Box)$ denotes the number of the column which contain our box. For some reason, explained below, we start the numeration with zero.

The collection of boxes situated to the right or below of $\Box$, including the box itself, we call a **hook** of $\Box$ and the number of boxes in it is denoted by $h(\Box)$. We have $h(\Box) = \lambda_i - i(\Box) + \lambda'_j - j(\Box) - 1$.

Theorem 6. In terms of notations introduced above, we have

$$P_\lambda(t) = \prod_{\Box \in D} \frac{t^{i(\Box)}}{1 - t^{h(\Box)}}. \tag{30}$$

A small disadvantage of this beautiful formula is the absence of symmetry between $i(\Box)$ and $j(\Box)$. It turns out that the symmetry can be achieved if we consider more general problem in the next section.

Here we want to introduce one more notion. Let G be a subgroup of $O(n,\,\mathbb{R})$. Denote by P the algebra of all polynomial functions on $\mathbb{R}^n$ with real coefficients and by I the subalgebra of G-invariant polynomials. Denote by ∂_i the operator of partial derivation $\partial/\partial x_i$. The formula

$$(p_1,\, p_2) := p_1(\partial_1,\, \partial_2,\, \dots,\, \partial_n) p_2 \,\big|_{x_1 = x_2 = \cdots = x_n = 0} \tag{31}$$

defines on P a structure of a Euclidean space.

The set of monomials $\{x^k := x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}\}$ forms an orthogonal basis in P with relations $(x^k,\, x^l) = \delta_{k,l} k!$ where $k! := (k_1)!(k_2)! \cdots (k_n)!$.

Let I_+ denote the ideal in I consisting of polynomials vanishing at the origin and let J denote the ideal in P generated by I_+. We denote by H the set of **harmonic polynomials** h satisfying

$$p(\partial)h = 0 \quad \text{for all} \quad p \in I_+. \tag{32}$$

Lemma 6. The orthogonal complement to J in P coincide with H.

$\Box$

Theorem 7. We have $P = I \cdot H$, i.e. every polynomial $p \in P$ can be written in the form

$$p = \sum_{k=1}^{N} p_k \cdot h_k \quad \text{where} \quad p_k \in I,\, h_k \in H. \tag{33}$$

Proof. Apply the induction on the degree d of p. For $d = 0$ we have $p = const \in I$. Assume that the theorem is true for all $d < d_0$ and consider a polynomial p of degree d_0. From the Lemma 6 we conclude that $p = q + h$ where $q \in J$ and $h \in H$. Hence, $q = \sum_k p_k \cdot q_k$ where $p_k \in I$, $q_k \in P$. Since degrees of q_k are less than degree of p, we can write $q_i = \sum_j p_{i,j} \cdot h_{i,j}$ with $p_{i,j} \in I$ and $h_{i,j} \in H$. Then $p = h + \sum_{i,j} p_k \cdot p_{i,j} \cdot h_{i,j}$ and we are done.

$\Box$

For the symmetric group S_n which is embedded in $O(n,\,\mathbb{R})$ as the group of permutation matrices, the stronger result holds:

Chevalley Theorem. The presentation (33) is unique. Hence, $P = I \otimes H$.
$\square$

12. Supersymmeric generalization.

Besides the polynomial algebra $\mathbb{C}[x_1, \ldots, x_n]$ we can consider its **odd analogue**, the Grassmann algebra $\wedge[\xi_1, \ldots, \xi_n]$ with generators satisfying the **anticommutation relations**:

$$\xi_k\xi_j + \xi_j\xi_k = 0 \quad \text{for all} \quad j,\, k. \tag{34}$$

It is a finite-dimensional graded algebra and dim $\wedge^k[\xi_1, \ldots, \xi_n] = \binom{n}{k}$.
Together with the polynomial algebra they generate so-called **Weil algebra**

$$E_n = \mathbb{C}[x_1, \ldots, x_n] \otimes \wedge[\xi_1, \ldots, \xi_n]. \tag{35}$$

The symmetric group S_n acts on $\wedge[\xi_1, \ldots, \xi_n]$ by permutation of generators and on E by simultaneous permutations of x_k and ξ_k. Sometimes, one identifies the variables ξ_k with differentials dx_k and obtains an isomorphism of E with an algebra of differential forms on $\mathbb{R}^n$ with polynomial coefficients.
The intertwining numbers $i(\pi_\lambda, \wedge[\xi_1, \ldots, \xi_n])$ and $i(\pi_\lambda, E$ are rather interesting. Below we shall describe explicitly their generating functions.

It turns out that for the algebra $\wedge[\xi_1, \ldots, \xi_n]$ the analog of Chevalley theorem holds: the whole algebra is isomorphic to the tensor product of the subalgebra of invariants and the subspace of harmonic elements.
Moreover, the algebra of invariants has dimension 2 and spanned by a constant 1 and the element $\Xi := \sum_k \xi_k$. The harmonic elements are those which satisfy $\sum_k \partial_k p = 0$ where ∂_k is the derivative with respect to the odd variable ξ_k defined by the formula

$$\partial_k\left(\xi_{k_1}\xi_{k_2}\cdots\xi_{k_m}\right) = \begin{cases} (-1)^{l-1}\xi_{k_1}\cdots\widehat{\xi_{k_l}}\cdots\xi_{k_m} & \text{if} \quad k_l = k \\ 0 \quad \text{if no} \quad k_l = k. \end{cases}$$

Exercise 12. Show that
a) $\wedge^k\langle\xi_1, \ldots, \xi_n\rangle = \Xi \cdot \wedge^{k-1}\langle\xi_1, \ldots, \xi_n\rangle \oplus H^k$
b) The representation of S_n in H^k is irreducible and corresponds to a hook diagram $(n-k, k)$.

From this exercise and formula (30) one derive the explicit formula for the multiplicities of π_λ in the bi-homogeneous components of the Weil algebra. Namely, let $m_{k,l}$ be the multiplicity of π_λ in $\mathbb{R}[x_1, \ldots, x_n \otimes \wedge^l\langle\xi_1, \ldots, \xi_n\rangle$. Then

$$P_\lambda(t,\, s) := \sum_{k,l} m_{k,l} \cdot t^k s^l = \prod_{\square\in D_\lambda} \frac{t^{i(\square)} + st^{j(\square)}}{1 - t^{h(\square)}}. \tag{36}$$

This formula for $s = 0$ coincides with (30). Substitution $t = 0$ is a bit more delicate and we leave to a reader to obtain from (36) the odd analog of (30).

13. Partition types in $T_n(\mathbb{F}_q)$.

Consider the distribution Let $T_n(\mathbb{F}_q)$ be the set of strictly upper triangular matrices over a finite field $\mathbb{F}_q$. Consider the distribution of these matrices according to their partition types (= Jordan normal forms). We say, that a matrix X has a **partition type** $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0)$ if it is conjugate in $GL(N, \mathbb{F}_q)$ to a direct sum $J_{\lambda_1} \oplus J_{\lambda_1} \oplus \cdots \oplus J_{\lambda_r}$ where J_k is a Jordan block of size k with the eigenvalue 0. Actually, it is more convenient to pass to the dual partition λ' and write it in the form $1^{\alpha_1} 2^{\alpha_2} \cdots n^{\alpha_n}$ where $\alpha_k = \lambda'_k - \lambda'_{k+1}$.

Denote by $Q_\alpha(q)$ the number of matrices $X \in T_n(\mathbb{F}_q)$, $n = \sum_{k=1}^n \alpha_k$, which have the dual partition type $1^{\alpha_1} 2^{\alpha_2} \cdots n^{\alpha_n}$. Writing the index $\alpha = \{\alpha_1,\, \alpha_2,\, \ldots,\, \alpha_n\}$ we omit all zeros in the end.

The initial polynomials look like

(37)
$$\begin{gathered} Q_1(q) = 1; \\ Q_2(q) = 1, \qquad Q_{0,1}(q) = q - 1; \\ Q_3(q) = 1, \qquad Q_{1,1}(q) = (q-1)(2q+1), \qquad Q_{0,0,1}(q) = q(q-1)^2; \\ Q_4 = 1, \quad Q_{2,1} = (q-1)(3q^2+2q+1), \quad Q_{0,2} = q(q-1)^2(2q+1), \\ Q_{1,0,1}(q) = q^2(q-1)^2(3q+1), \quad Q_{0,0,0,1} = q^3(q-1)^3. \end{gathered}$$

It suggest that polynomials Q_α satisfy a recurrence relations of the form

$$Q_\alpha = \sum_k m(k,\, \alpha\,;\, q) \cdot Q_{\alpha - \delta_k} \tag{38}$$

where $\alpha - \delta_k = (\alpha_1,\, \ldots,\, \alpha_{k-1} + 1,\, \alpha_k - 1, \alpha_{k+1}, \ldots, \alpha_n)$ and k runs through the values for which $\alpha_k > 0$.

The precise value of of coefficients in (38) was found recently by Aaron Smith

$$m(k,\, \alpha;\, q) = \begin{cases} q^{n-1-\sum_{i\geq k}\alpha_i} - q^{n-1-\sum_{i\geq k-1}\alpha_i} & \text{for} \quad k > 1 \\ q^{n-1-\sum_{i\geq 1}\alpha_i} \quad \text{for} \quad k = 1. \end{cases} \tag{39}$$

These relation open the way to explicit computation of all Q_α but now only partial results are know. E.g., for so-called **hook diagrams** when $\lambda = (m+1, \underbrace{1,\, 1, \ldots,\, 1}_{k \quad \text{times}})$ and $\lambda' = (k+1, \underbrace{1, 1, \ldots, 1}_{m \quad \text{times}})$ with $\alpha_1 = m,\ \alpha_2 = \cdots = \alpha_k = 0,\ \alpha_{k+1} = 1$, we have

$$Q_{m,\underbrace{0,\, 0, \ldots,\, 0}_{k-1 \quad \text{zeros}},1} = (q-1)^k q^{\frac{(k-1)(2m+k)}{2}} \sum_{j=0}^m \binom{k+j}{j} q^j. \tag{40}$$

Theorem 8. The polynomial Q_α has degree $\sum_{i<j} \lambda'_i \lambda'_j$ and the form

$$Q_\alpha(q) = (q-1)^{n-\lambda'_1} q^a(\alpha) Q'_\alpha(q)$$

where $Q'(q)$ is some polynomial of degree $b(\alpha)$ such that $Q'_\alpha(0) \neq 0$ and $Q'_\alpha(1) \neq 0$.

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Department of Mathematics, The University of Pennsylvania, Philadelphia, PA 19104-6395 *E-mail address: kirillov@math.upenn.edu*