I am interested in algebraic geometry and mathematical physics. My research is about constructing singular monopoles on odd dimensional manifolds that admit a circle fibration. The idea is to apply Fourier-Mukai transform along the circle direction to change the problem into finding holomorphic line bundles satisfying certain conditions. This work is an extension and generalization of [CH11], and is inspired by [DP03].

Monopole equations

Being hypothetical particles, monopoles have been fascinating researchers for years. The introduction of monopoles not only leads to a symmetric form of Maxwell's equations, but also gives rise to a natural quantization of electric charges. Dirac monopoles, as described in [Dir31], are a class of singular solutions to Maxwell's equation in $\mathbb{R}^3$. To extend the study of monopoles to an arbitrary Riemannian 3-manifold X, we adopt the formalism of field theory, which describes a monopole as a triple (E, A, ϕ) on X, where E is a Hermitian vector bundle on X, A is a unitary connection on E, and ϕ is a Higgs field, i.e. a skew-Hermitian section of $\mathrm{End}(E)$. To constitute a monopole, together these data should satisfy Bogomolny's equation

$$F_A = *d_A\phi$$

When X is fibered over a Riemann surface Σ, we have the pullback ω_Σ of the Kähler form on Σ, and can consider the generalized Bogomolny equation:

$$F_A - \sqrt{-1}C\mathrm{id}_E\omega_\Sigma = *d_A\phi \tag{1}$$

where C is a real constant. As Bogomolny's equation comes from dimensional reduction of Yang-Mills equation, the generalized Bogomolny equation comes from dimensional reduction of the Hermitian-Yang-Mills equation, and hence people often refer to it as Hermitian-Bogomolny equation and to its solution as Hermite-Einstein monopoles.

Hermite-Einstein monopoles on $[0, 1] \times \Sigma$ play an important role in Kapustin-Witten's field theory approach [KW06] to the geometric Langlans program. More recently, in a similar context, Charbonneau and Hurtubise studied in [CH11] singular Hermite-Einstein monopoles on $S^1 \times \Sigma$, and argued that such monopoles are in one-to-one correspondence with certain meromorphic Higgs bundles over Σ satisfying a stability condition. This converts the existence problem for monopoles to a purely algebraic-geometric problem and allows for construction of monopoles by projective geometry techniques. In my thesis, I extend and generalize the Charbonneau-Hurtubise correspondence to give an algebraic geometric construction of singular monopoles over odd dimensional compact manifolds which are the total spaces of non-trivial S^1 bundles over a complex variety.

To understand how the monopole equation (1) generalizes to higher dimensions, we can apply the dimensional reduction argument. Let X be a principal S^1 bundle over Σ, where Σ is a compact complex manifold of dimension k, and let Y be the associated $\mathbb{C}^*$ bundle over Σ. In local coordinates $(t, z_1, \cdots, z_k, \bar{z}_1, \cdots, \bar{z}_k)$ of X, we can identify $\mathbb{C}^*$ with $\mathbb{R}^1 \times S^1$, and parameterize the radial direction by r. Given a vector bundle $\widetilde{E}$ on Y with a connection $\widetilde{A} = A_r \mathrm{d}r + A_t \mathrm{d}t + \sum_{j=1}^{k}(A_k \mathrm{d}z_k + A_{\bar{k}} \mathrm{d}z_{\bar{k}})$, by identifying $r + \sqrt{-1}t$ with a complex coordinate z_0, we can write the connection as $\widetilde{A} = \sum_{j=0}^{k}(A_k \mathrm{d}z_k + A_{\bar{k}} \mathrm{d}z_{\bar{k}})$, and we say that $\widetilde{A}$ satisfies Hermitian-Yang-Mills equation if

$$\widetilde{F}^{0,2} = \widetilde{F}^{2,0} = 0, \quad g^{j\bar{l}} \widetilde{F}_{j\bar{l}} = \sqrt{-1} C \mathrm{id}, \quad 0 \leq j, l \leq k,$$

where $\widetilde{F}$ denotes the curvature of $\widetilde{A}$; $\widetilde{F}^{0,2}$ and $\widetilde{F}^{2,0}$ are curvature forms in the corresponding type decomposition; $\widetilde{F}_{j\bar{l}} = \partial_j A_{\bar{l}} - \bar{\partial}_l A_j + [A_j, A_{\bar{l}}]$; and g is the product Käher metric on Y. If the connection doesn't depend on r, we can reduce the dimension down by one, and the equations can be written as

$$\begin{cases} F_{jl} = 0, \quad F_{\bar{j}\bar{l}} = 0 \\ F_{tj} = \sqrt{-1} \nabla_j A_r, \quad F_{t\bar{j}} = -\sqrt{-1} \nabla_{\bar{j}} A_r \\ -\sqrt{-1} \nabla_t A_r + g^{j\bar{l}} F_{j\bar{l}} = \sqrt{-1} C I \end{cases} \tag{2}$$

where F denotes the curvature of $A = A_t \mathrm{d}t + \sum_{j=1}^{k}(A_k \mathrm{d}z_k + A_{\bar{k}} \mathrm{d}z_{\bar{k}})$, $\nabla_j := (\partial_j + A_j)\mathrm{d}z_j$, $\nabla_{\bar{j}} := (\bar{\partial}_j + A_{\bar{j}})\mathrm{d}\bar{z}_j$, and $\nabla_t := (\partial_t + A_t)\mathrm{d}t$. Thus we can think that these equations are about a connection A, and a Higgs field $\phi = A_r$. In three dimension, this is equivalent to the Hermitian-Bogomolny equation, and therefore we take (2) as generalized monopole equation.

Monopoles on $S^1 \times \Sigma$

Let Σ be a Riemann surface, Charbonneau and Hurtubise consider monopoles with

Dirac-type singularities, which means that near a singular point,

$$\phi = \frac{\sqrt{-1}}{2R}\mathrm{diag}(l_1, l_2, \cdots, l_n) + O(1), \quad \mathrm{d}_A(R\phi) = O(1),$$

where R is the distance to the singular point. When (A, ϕ) have these asymptotics, we say that the singularity of the monopole has type $\vec{l} = (l_1, l_2, \cdots, l_n)$. Now parameterize $S^1 = \mathbb{R}/\mathbb{Z}$ by $t \in [0, 1]$, and consider monopole (E, A, ϕ) with singularities at $p_j = (t_j, z_j) \in S^1 \times \Sigma$ of types $\vec{k_j} = (k_{j1}, k_{j2}, \cdots, k_{jn})$, $j = 1, 2, \cdots, N$. Out of these data Charbonneau and Hurtubise construct a Higgs pair $(\mathcal{E}, \rho)$ by setting $\mathcal{E} = E|_{\{0\}\times\Sigma}$, and letting ρ be the parallel transport with respect to $\nabla_t - \sqrt{-1}\phi$. The map $\rho : \mathcal{E} \to \mathcal{E}$ is a meromorphic bundle map, which is invertible and holomorphic away from z_j's, and near the z_j has the form

$$\rho(z) = F(z)\mathrm{diag}((z - z_j)^{k_{j1}}, \cdots, (z - z_j)^{k_{jn}})G(z)$$

with $F(z)$ and $G(z)$ holomorphic and invertible.

In general, consider an arbitrary pair $(\mathcal{E}, \rho)$ consisting of a holomorphic vector bundle $\mathcal{E}$ over Σ, and a meromorphic bundle map $\rho : \mathcal{E} \to \mathcal{E}$ with singularities on z_j, $j = 1, 2, \cdots, N$. Iwahori et al. prove in [IM65] that around each z_j, ρ can always be expressed as the above form, and we say that the bundle pair has singular type $((z_1, \vec{k_1}), \cdots, (z_N, \vec{k_N}))$. Given a list of numbers $\vec{t} = (t_1, t_2, \cdots, t_N) \in T^N$, where T^N denotes the torus formed by the product of N circles of circumference 1, define the $\vec{t}-$degree of the bundle pair $(\mathcal{E}, \rho)$ as

$$\delta_{\vec{t}}(\mathcal{E}, \rho) = c_1(\mathcal{E}) - \sum_{j=1}^{N} t_j(k_{j1} + k_{j2} + \cdots + k_{jn})$$

and define its $\vec{t}$-slope as

$$\mu_{\vec{t}}(\mathcal{E}, \rho) = \delta_{\vec{t}}(\mathcal{E}, \rho)/\mathrm{rank}\ \mathcal{E}$$

A bundle pair is $\vec{t}$-stable if any proper non-trivial ρ-invariant subbundle has a strictly smaller $\vec{t}$-slope. Charbonneau and Hurtubise prove that under the condition of $\sum k_{jl} = 0$ (which is necessary for singular monopoles to exist), there is an equivalence between

$$\left\{\begin{matrix}\text{Irreducible } U(n) \text{ monopoles } (E, A, \phi) \text{ on } S^1 \times \Sigma \text{ with} \\ \text{Dirac type singularities at } (t_j, z_j) \text{ of singular type } \vec{k}_j\end{matrix}\right\} \tag{3}$$

and

$$\left\{\begin{array}{l}\vec{t}-\text{stable bundle pairs } (\mathcal{E},\rho) \text{ on } \Sigma, \text{ with } \mathcal{E} \\ \text{of rank } n \text{ and } \rho \text{ singular at } z_j \text{ of type } \vec{k}_j\end{array}\right\} \tag{4}$$

Spectral data and the Fourier-Mukai transform

To generalize this result, observe that with any bundle pair $(\mathcal{E},\rho)$ over Riemann surface Σ, we can associate spectral data $(\widetilde{\Sigma},\mathcal{L})$. Here $\widetilde{\Sigma}\subset\Sigma\times\mathbb{C}^*$ is the branched cover of $\Sigma-\{z_1,\cdots,z_N\}$, parametrizing the eigenvalues of $\rho(z)$ as z varies in $\Sigma-\{z_1,\cdots,z_N\}$, and $\mathcal{L}$ is the subsheaf of $p^*\mathcal{E}$, which assigns to (z,λ) the eigenspace $\ker(p^*\rho(z)-\lambda\cdot\mathrm{id})$, where $p:\Sigma\times\mathbb{C}^*\to\Sigma$ is the projection to Σ. Assuming $\rho(z)$ is regular everywhere (i.e. that it has distinct eigenvalues for different Jordan blocks), then $\mathcal{L}$ is a line bundle on $\widetilde{\Sigma}$. Note that being nonregular is a complex codimension three condition, and as in the case of Riemann surface, a generic ρ is regular.

The spectral data $(\widetilde{\Sigma},\mathcal{L})$ reconstructs the meromorphic Higgs pair $(\mathcal{E},\rho)$. Indeed, if we denote the restriction of the projection p to $\widetilde{\Sigma}$ by $\pi:\widetilde{\Sigma}\to\Sigma$, then we have $\mathcal{E}=\pi_*\mathcal{L}$ on $\Sigma-\{z_1,\cdots,z_N\}$, and $\rho=\pi_*\eta$, where η is the tautological section of $p^*(\Sigma\times\mathbb{C}^*)\simeq\Sigma\times\mathbb{C}^*\times\mathbb{C}^*$, consisting of points (z,λ,λ) in local coordinates. In this way, we can recover the bundle pair from its spectral data and therefore get a one-to-one correspondence.

Next we interpret stability as numerics controlling singularities of the spectral data. Near a singular point z_j, $\rho(z)$ is asymptotic to $\mathrm{diag}((z-z_j)^{k_{j1}},(z-z_j)^{k_{j2}},\cdots,(z-z_j)^{k_{jn}})$, thus the spectral cover has several branches approaching to 0 or to ∞, and the growth rate is controlled by the power k_{jl}.

Given an n-sheeted branched cover $\widetilde{\Sigma}$ of $\Sigma-\{z_1,\cdots,z_N\}$, in a small annulus neighborhood of z_j, we have n disjoint sheets covering it, and we say that the cover has singular type $\vec{k}_j=(k_{j1},\cdots,k_{jn})$ if the leading term of Laurent expansion for the l-th sheet is $z^{k_{jl}}$. Then given a vector $\vec{t}=(t_1,\cdots,t_N)\in T^N$, we can define the $\vec{t}$-degree of the spectral datum $(\widetilde{\Sigma},\mathcal{L})$ as

$$\delta_{\vec{t}}=c_1(\mathcal{L})-\sum_{j=1}^{N}t_j(k_{j1}+k_{j2}+\cdots+k_{jn})$$

and its $\vec{t}$-slope as $\delta_{\vec{t}}/n$. A spectral datum $(\widetilde{\Sigma},\mathcal{L})$ is $\vec{t}$-stable if for any subsheaf $\mathcal{F}\subset\imath_{\widetilde{\Sigma}*}\mathcal{L}$

of pure dimension one the $\vec{t}$-slope of $\mathcal{F}$ is strictly smaller than that of $(\widetilde{\Sigma}, \mathcal{L})$.

Therefore we have an equivalence between (4) and

$$\left\{ \begin{array}{l} \vec{t}-\text{stable spectral data } (\widetilde{\Sigma}, \mathcal{L}),\ \text{where } \widetilde{\Sigma} \text{ is an } n \text{ sheeted cover of } \Sigma,\ \text{with} \\ \text{singular type } \vec{k}_j \text{ over } z_j,\ \text{and } \mathcal{L} \text{ is in the Prym variety of } p : \widetilde{\Sigma} \to \Sigma \end{array} \right\} \quad (5)$$

So in the language of spectral data, we can restate the Charbonneau-Hurtubise theorem as the equivalence (3) $\iff$ (5).

The starting point of my project is the observation that the correspondence (3) $\iff$ (5) can be constructed directly, by T-dualizing the circle fiber direction. This gives the correct picture because T-duality identifies {vector bundle of rank n with a flat connection along circle direction} with { n points on the dual circle and a complex line assigned to each point}. In our case, we can realize T-duality explicitly via a Fourier-Mukai transform: consider the Poincaré line bundle $\mathscr{P}$ on $S^1 \times \Sigma \times \mathbb{C}^*$, we can construct a connection ∇ which is flat in $\mathbb{C}^*$ direction, such that Fourier-Mukai transform with kernel $(\mathscr{P}, \nabla)$ identifies monopole data $(E, A - \sqrt{-1}\phi)$ with the spectral data $(\widetilde{\Sigma}, \mathcal{L})$.

Furthermore, we can apply the Fourier-Mukai transform to non-trivial S^1-bundles, which allows us to extend Charbonneau and Hurtubise's work to a general S^1 fibered spaces.

Monopoles on S^1-fibered manifold

For concreteness consider first the 3-dimensional case. Let X be a principal S^1 bundle over a Riemann surface Σ, and let (E, A, ϕ) be a singular monopole with singularities located over $z_1, \cdots, z_N \in \Sigma$. Take an open cover $\{U_i\}$ of Σ, so that for each U_i the circle bundle $X_i := X_{|U_i} \to U_i$ trivializes. Using the trivialization we can construct a normalized Poincare line bundle with connection $(\mathscr{P}_i, \nabla_i)$ on $S^1 \times U_i \times \mathbb{C}^*$. The Fourier-Mukai transform with kernel $(\mathscr{P}_i, \nabla_i)$ converts the monopole $(E, A, \phi)_{|X_i}$ into spectral data on $U_i \times \mathbb{C}^*$. The spaces $U_i \times \mathbb{C}^*$ glue into $\Sigma \times \mathbb{C}^*$, but the Poincare line bundles do not glue into a global line bundle with connection on $\Sigma \times \mathbb{C}^*$. The obstruction to gluing the $(\mathscr{P}_i, \nabla_i)$'s is a smooth $U(1)$ gerbe with Chern connection on $\Sigma \times \mathbb{C}^*$. This gerbe is classified by the cohomology class

$$[H] = [p_\Sigma^* \omega \wedge p_{\mathbb{C}^*}(\mathrm{d}z/z + \mathrm{d}\bar{z}/\bar{z})] \in H^3(\Sigma \times \mathbb{C}^*; \mathbb{Z}),$$

where ω denotes the Kähler form on Σ. Since the curvature of this gerbe is of type $(2,1)+(1,2)$ we can recast it as complex geometric data, i.e. as a holomorphic $\mathcal{O}^\times$-gerbe $\mathcal{H}$ over $\Sigma \times \mathbb{C}^*$.

The locally defined Poincare line bundles with connections will glue into a global pair $(\mathscr{P}, \nabla)$ on the gerbe $X \times_\Sigma \mathcal{H}$. The Fourier-Mukai transform associated with $(\mathscr{P}, \nabla)$ takes the monopole data $(E, A - \sqrt{-1}\phi)$ on X to spectral data $(\tilde{\Sigma}, \mathcal{L})$, where $\widetilde{\Sigma}$ is an n-sheet branch cover of $\Sigma - \{z_1, \cdots, z_N\}$, and $\mathcal{L}$ is a weight one line bundle on the restriction of $\mathcal{H}$ to $\widetilde{\Sigma}$, or equivalently a trivialization of the gerbe $\mathcal{H}_{|\widetilde{\Sigma}}$. The irreducible monopoles will correspond to spectral data satisfying a suitable stability condition, and we conjecture that $\vec{t}$-stable is still the right condition. Moreover spectral data $(\widetilde{\Sigma}, \mathcal{L})$ for which the spectral curve $\widetilde{\Sigma}$ is smooth and irreducible will be automatically stable. This gives a concrete description of a dense Zariski open set of the moduli of singular monopoles.

More formally, denote by $\mathcal{M}_n(X, z_1, \cdots, z_N, \vec{k}_1, \cdots, \vec{k}_N)$ the moduli of irreducible $U(n)$ monopoles on X with singularities over $z_j \in \Sigma$ of type $\vec{k}_j$, $1 \leq j \leq N$. Let

$$h: \ \mathcal{M}_n(X, z_1, \cdots, z_N, \vec{k}_1, \cdots, \vec{k}_N) \to B$$

be the map sending a monopole (E, A, ϕ) to the closure of the spectral cover $\widetilde{\Sigma} \subset \Sigma \times \mathbb{C}^*$ inside $\Sigma \times \mathbb{P}^1$. The target B of this map is a linear system of curves in $\Sigma \times \mathbb{P}^1$ with prescribed order of contact with $\Sigma \times \{0\}$ and $\Sigma \times \{\infty\}$ over the points $z_j \in \Sigma$. The fiber of h over a smooth and irreducible spectral curve will be a torsor over the Prym variety $\mathrm{Prym}(\widetilde{\Sigma}/\Sigma)$ and the corresponding spectral pairs will all be stable. In particular, we can reverse this process and use it to construct monopoles. To obtain a singular monopole on X, with Dirac type singularities located over $z_1, \cdots, z_N \in \Sigma$, of singular type $\vec{k}_j$, we start with a smooth divisor $\widetilde{\Sigma} \subset \Sigma \times \mathbb{C}^*$ which intersects $\Sigma \times \{0\}$ and $\Sigma \times \{\infty\}$ at the points $(z_j, 0)$ and (z_j, ∞) with multiplicities given by the $\vec{k}_j$'s. We can always trivialize the gerbe $\mathcal{H}$ on $\widetilde{\Sigma}$, since any 3-form vanishes on a curve. Choose some trivialization $\mathcal{A}$ of the gerbe $\mathcal{H}_{|\widetilde{\Sigma}}$. Then for any line bundle $L \in \mathrm{Prym}(\widetilde{\Sigma}/\Sigma)$ the product $\mathcal{A} \otimes L$ will be a line bundle on $\mathcal{H}_{|\widetilde{\Sigma}}$ and if we apply the Fourier-Mukai transform to this line bundle we will get a complex vector bundle E with connection ∇ on X. If we choose a unitary connection A on E and subtract it from ∇ we will get a natural Higgs field $\phi = -\sqrt{-1}(\nabla - A)$. When $(\widetilde{\Sigma}, \mathcal{A} \otimes L)$ satisfies the correct stability condition, we expect that there exists a unique unitary connection A on E so that the triple (E, A, ϕ) is a monopole. This is a twisted monopole version of the Kobayashi-Hitchin correspondence.

The methods discussed above work on higher dimensions as well, if we require Σ to be a k-dimensional compact complex manifold, and a singular monopole (E, A, ϕ) satisfying the generalized Hermitian-Bogomolny equations (2), with singularities located on N disjoint compact real codimension 3 submanifolds $C_1, \cdots, C_N$, which project isomorphically to smooth complex hypersurfaces in Σ, and such that at every point of C_j, the Higgs field ϕ restricted to normal direction of C_j is asymptotically $\frac{\sqrt{-1}}{2R}\mathrm{diag}(k_{j1}, \cdots, k_{jn})$. Again in this case we can use Fourier-Mukai transform along the fiber direction to get spectral data $(\widetilde{\Sigma}, \mathcal{L})$, with $\widetilde{\Sigma} \subset \Sigma \times \mathbb{C}^\times$ an n sheet cover of Σ and $\mathcal{L}$ a trivialization of the gerbe restricted to $\widetilde{\Sigma}$. With a suitable stability condition, we can identify singular monopoles with stable spectral data as before. Again we expect that $\vec{t}$-stability is the correct condition in this setting.

Future plans

In my thesis I used projective geometry to construct twisted spectral data on products of complex manifolds with $\mathbb{C}^*$ and studied the stability and moduli spaces of such data. I also constructed a specific holomorphic-to-semi-flat Fourier-Mukai transform that produces monopole data when it is applied to the twisted spectral data. The fact that this duality construction agrees with the Charbonneau-Hurtubise correspondence in the untwisted case suggests that the monopole version of the Kobayashi-Hitchin correspondence holds and the monopole connection obtained as the Fourier-Mukai image of stable spectral data can be split uniquely into a unitary connection and a Higgs field that obey the Hermitian Bogomolony equations.

I have checked that the Kobayashi-Hitchin correspondence holds also in the twisted context when Σ is an abelian variety. I am currently working on proving this twisted correspondence for a general Σ. One way to tackle this problem is is to reduce it to a twisted version of the Corlette-Simpson theorem. Notice that after the Fourier-Mukai transform, the Hermitian-Bogomolny equation becomes a differential equation involving a twisted meromorphic Higgs bundle on the Σ. I plan to use a twisted version of non-abelian Hodge theory to interpret the existence of solution of this twisted Yang-Mills-Higgs equation as a stability condition. This will convert the problem in to the purely algebraic-geometric question of comparing stability for twisted Higgs bundles with stability of the associated spectral data. Hopefully the spectral stability arising in this way agrees with $\vec{t}$-stability. I plan to investigate this relationship of stabilities in detail.

A different and important issue is to understand the extendability properties of the gerbe $\mathcal{H}$. This gerbe is defined on $\Sigma \times \mathbb{C}^*$ it is likely that it can be extended to a gerbe on $\Sigma \times \mathbb{P}^1$ or some birational modification of this product. When this happens we can consider the compactified spectral cover describe the stability condition as a stability condition for parabolic structures attached to the compactified spectral data. I will study systematically the possible birational extensions of holomorphic gerbes and the moduli problems of parabolic sheaves on such gerbes.

Finally with the correct stability condition, I can study the moduli space of monopoles by classifying stable spectral data, and I am eager to see how it depends on the geometry and topology of the space. In particular for Σ of higher dimension not every spectral cover will support twisted spectral data. The members of the linear system of spectral covers, that support twisted spectral data have to obey a non-trivial constraint of Noether-Lefschetz type. I plan to study the loci of such divisors and the geometric properties of the resulting components of the moduli of monopoles. I have already obtained expected dimension formulas and plan to investigate the question of existence and smoothness of the moduli space.

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