

Self-dual and quasi self-dual algebras

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A *self-dual* algebra is an associative or Lie algebra A together with an A bimodule isomorphism $A \rightarrow A^{\vee\text{op}}$, where $A^{\vee} = \text{Hom}_k(A, k)$, the dual bimodule to A (considered as an A bimodule), and $A^{\vee\text{op}}$ is the the same underlying k module as A^{\vee} but is an A bimodule whose left operation by an element $a \in A$ is the same as the right operation by a on A^{\vee} , and similarly with left and right interchanged. This induces an isomorphism $H^*(A, A) \cong H^*(A, A^{\vee\text{op}})$; algebras with such an isomorphism are *quasi self-dual*. For these algebras $H^*(A, A)$ is a contravariant functor of A . They form a full subcategory of the category of associative or Lie algebras, respectively. Finite dimensional associative self-dual algebras over a field are identical with symmetric Frobenius algebras (which are closely connected to 1+1 dimensional topological quantum field theory). Finite poset algebras are quasi self-dual.

For an important class of algebras A the cohomology $H^*(A, A)$ of A with coefficients in itself is, remarkably, a contravariant functor of A . Many of these algebras are related to geometric objects.

It is a familiar fact that for most categories of geometric objects cohomology is a contravariant functor. The cohomology of algebras behaves somewhat perversely. If A is an associative algebra over a (commutative, associative, unital) ring k and M an A -bimodule then the Hochschild cohomology $H^*(A, M)$ is a contravariant functor of A but a covariant functor of M , and similarly with the Eilenberg-Chevalley cohomology of Lie algebras (but note that the first instance of Lie cohomology with coefficients in a non-trivial module seems to be due, again, to Hochschild). One of the most important A -bimodules is A itself, for $H^*(A, A)$ has a rich structure and governs in particular the deformations of A , but in general $H^*(A, A)$ is neither a covariant nor contravariant functor of A . For an important class of algebras, however, $H^*(A, A)$ is indeed a contravariant functor of A . These are the *quasi self-dual* ones, those for which there is an isomorphism $H^*(A, A) \cong H^*(A, A^{\vee\text{op}})$. We consider here only algebras A which are finite free modules over k , and likewise for A bimodules, but some of the most interesting cases are likely to be infinite dimensional, requiring topological considerations.

For categories of algebras where it is meaningful to consider the cohomology $H^*(A, k)$ of A with coefficients in k as a trivial module this cohomology, which

frequently has a geometric interpretation, is generally contravariant. For example, let G be a finite group and denote the group ring of a group G by kG ; this operates on k by setting $gc = c = cg = c$ for every $g \in G, c \in k$. If \mathcal{S} is any contractible space on which G operates freely then the Hochschild cohomology $H^*(kG, k)$ is naturally isomorphic with the cohomology of the quotient space \mathcal{S}/G with coefficients in k , and the latter is contravariant. However, $H^*(kG, kG)$ does not have such a simple geometric interpretation, and while $H^*(kG, kG)$ and $H^*(kG, k)$ are both in a natural way rings, there does not seem to be any simple way to deduce the structure of the former from the latter. Nevertheless finite group rings are *self-dual*; these are the algebras A equipped with an isomorphism $A \rightarrow A^{\text{vop}}$. (The isomorphism need not be unique but in some cases there is a natural choice.) This raises the question of whether there is still an underlying geometric explanation, and more generally, if there is an underlying geometric interpretation for any quasi self-dual algebra. Another basic question not considered here is, When does a morphism $B \rightarrow A$ of quasi self-dual algebras induce a morphism $H^*(A, A) \rightarrow H^*(B, B)$ not merely of k modules but one which preserves additional structures which $H^*(A, A)$ may possess such as, in the associative case, the Gerstenhaber algebra structure.

The following problem motivated this paper. To every finite poset (partially ordered set) \mathcal{P} , which we may view as a small category, one can associate a k -algebra A such that $H^*(A, A)$ is canonically isomorphic to the simplicial cohomology of the nerve of \mathcal{P} with coefficients in k ; the proof will be revisited below. If we start with a simplicial complex Σ then its faces, ordered by setting $\sigma \preceq \tau$ whenever σ is a face of τ , form a poset whose nerve is the barycentric subdivision of Σ and which therefore has the same simplicial cohomology. (Amongst the faces of a simplex here one includes the simplex itself.) To every finite simplicial complex Σ one can therefore associate a k -algebra $A = A(\Sigma)$ such that $H^*(A, A) \cong H^*(\Sigma, k)$, where the left side is Hochschild cohomology and the right simplicial. (Simplicial cohomology is in fact a special case of Hochschild cohomology independent of any finiteness assumption, cf. [7].) Using the isomorphism with simplicial cohomology one sees that if A and B are poset algebras then a poset morphism $B \rightarrow A$ in fact induces a morphism $H^*(A, A) \rightarrow H^*(B, B)$ just because it induces a simplicial morphism of the associated simplicial complexes. The problem was to exhibit a purely algebraic reason for this functoriality. We show (by a simple extension of the proof of the preceding result) that poset algebras are in fact quasi self-dual. As a consequence it is not necessary to restrict to algebra morphisms induced by morphisms of the underlying posets. There are other morphisms between poset algebras but they are probably very restricted in nature.

1 Opposite algebras and modules.

The *opposite* of an associative k algebra A , denoted A^{op} , is the same underlying k -module but with reversed multiplication: $a \circ b \in A^{\text{op}}$ is defined to be the element $ba \in A$. (We will generally use “ \circ ” to denote an opposite multiplication.)

Left modules over A are right modules over A^{op} and an A -bimodule may be viewed as a left module over the universal enveloping algebra $A^e = A \otimes A^{\text{op}}$. An algebra morphism of the form $\phi : B \rightarrow A^{\text{op}}$ is sometimes called an antimorphism; it is a k -module mapping such that $\phi(ab) = \phi b \cdot \phi a$ (and is the same thing as a morphism from B^{op} to A). If $\phi : B \rightarrow A$ is an antimorphism and M an A -bimodule then the underlying k -module of M becomes a B^{op} -bimodule by setting $b \circ m \circ b' = \phi b' \cdot m \cdot \phi b$. The universal enveloping algebra A^e has an antiautomorphism interchanging the tensor factors. If G is a group, then its group algebra kG has an antiautomorphism σ sending $g \in G$ to g^{-1} , extended linearly. The infinitesimal version of this is that every Lie algebra \mathfrak{g} has an antiautomorphism sending $a \in \mathfrak{g}$ to $-a$.

The *opposite of an A -bimodule M* is the A^{op} bimodule M^{op} which is the same underlying k module but where we set $a \circ m \circ b \in M^{\text{op}}$ equal to $b m a \in M$. Note that M^{op} is an A^{op} bimodule. Although a commutative algebra A is identical with its opposite, a bimodule M over A is generally distinct from its opposite since the left and right operations of A may be distinct, as in e.g. A^e considered as an A -bimodule. If they are identical then M is called *symmetric*. Left modules over a commutative algebra may be viewed as symmetric bimodules.

Theorem 1 *Let M be an A bimodule and $F \in C^n(A, M)$ be a Hochschild n-cochain. Define $F^{\text{op}} \in C^n(A^{\text{op}}, M^{\text{op}})$ by $F^{\text{op}}(a_1, \dots, a_n) = F(a_n, \dots, a_1)$. The map $C^*(A, M) \rightarrow C^*(A^{\text{op}}, M^{\text{op}})$ defined by $F \mapsto (-1)^{\lfloor \frac{n+1}{2} \rfloor} F^{\text{op}}$ is a cochain isomorphism inducing an isomorphism of cohomology $H^*(A, M) \cong H^*(A^{\text{op}}, M^{\text{op}})$.*

PROOF. One has

$$\begin{aligned} \delta(F^{\text{op}})(a_1, \dots, a_{n+1}) &= \\ a_1 \circ F^{\text{op}}(a_2, \dots, a_{n+1}) - F^{\text{op}}(a_1 \circ a_2, \dots, a_{n+1}) + \dots + (-1)^{n+1} F^{\text{op}}(a_1, \dots, a_n) \circ a_{n+1} \\ &= F(a_{n+1}, \dots, a_2) a_1 - F(a_{n+1}, \dots, a_3, a_2 a_1) + \dots + (-1)^{n+1} a_{n+1} F(a_n, \dots, a_1) \\ &= (-1)^{n+1} (\delta F)^{\text{op}}(a_1, \dots, a_{n+1}). \end{aligned}$$

The introduction of the sign $(-1)^{\lfloor \frac{n+1}{2} \rfloor}$ just corrects for the sign $(-1)^{n+1}$ in the foregoing. \square

Similarly, if $\phi : B \rightarrow A$ is an antimorphism and if M is an A bimodule then the k -module morphism $\phi^* : C^*(A, M) \rightarrow C^*(B^{\text{op}}, M)$ sending $F \in C^n(A, M)$ to $\phi^n F \in C^n(B^{\text{op}}, M)$ defined by

$$(\phi^n F)(b_1, \dots, b_n) = (-1)^{\lfloor \frac{n+1}{2} \rfloor} \phi(F(\phi b_n, \dots, \phi b_1))$$

is a cochain morphism. Likewise, if M, N are A bimodules and $T : M \rightarrow N^{\text{op}}$ is an antimorphism then the k -module morphism $T^* : C^*(A, M) \rightarrow C^*(A^{\text{op}}, N)$ sending $F \in C^n(A, M)$ to $T^n F \in C^n(A^{\text{op}}, N)$ defined by

$$(T^n F)(a_1, \dots, a_n) = T(F(a_n, \dots, a_1))$$

is a cochain morphism. Therefore, with the preceding notation an antimorphism $\phi : B \rightarrow A$ induces a morphism of cohomology $H^*(A, M) \rightarrow H^*(B^{\text{op}}, M) \cong H^*(B, M^{\text{op}})$ and an antimorphism $T : M \rightarrow N^{\text{op}}$ induces a morphism $H^*(A, M) \rightarrow H^*(A^{\text{op}}, N^{\text{op}}) \cong H^*(A, N)$.

The next section introduces the dual, M^\vee of an A module M , which is an A^{op} module in a natural way, and the foregoing will imply that $H^*(A^{\text{op}}, M^\vee) \cong H^*(A, M^{\vee\text{op}})$. However, even when A is commutative and M a symmetric bimodule, $H^*(A, M)$ is generally *not* isomorphic to $H^*(A, M^\vee)$, as an example in the next section will show.

The dual of a Lie algebra \mathfrak{g} is similarly defined by $\mathfrak{g}^\vee = \text{Hom}(\mathfrak{g}, k)$ but in this case we may simply view \mathfrak{g}^\vee as a \mathfrak{g} module: if $c \in \mathfrak{g}$ and $f \in \mathfrak{g}^\vee$ then $[c, f]$ is defined by setting $[c, f](a) = -f([c, a])$ for all $a \in \mathfrak{g}$. (If V, W are modules over a Lie algebra \mathfrak{g} then $\text{Hom}_k(V, W)$ becomes a \mathfrak{g} module as follows: if $\phi : V \rightarrow W$ and $c \in \mathfrak{g}$ then $[c, \phi]$ is defined by setting $[c, \phi](a) = [c, \phi a] - \phi([c, a])$ for all $a \in \mathfrak{g}$. The preceding is the special case where $V = \mathfrak{g}, W = k$; note that the coefficient ring k is a Lie module over any Lie algebra \mathfrak{g} but the operation is trivial, so the first term on the right, which would be $[c, f(a)]$, vanishes.) Theorem 1 is not really necessary in the Lie case where cocycles are alternating. Sending a cocycle to its opposite leaves it unchanged if the dimension is even and just reverses the sign if it is odd.

2 Self-dual and Frobenius algebras

Suppose that A is an associative k -algebra and let M be an A -bimodule which (like A) will always be assumed to be free and of finite rank as a k module. Its *dual*, $M^\vee = \text{Hom}_k(M, k)$ is then again a free k module and of the same rank as M but should be viewed as an A^{op} -bimodule since the action of A is reversed: If $f \in M^\vee$ and $a, b, x \in A$ then $(a \cdot f \cdot b)(x) = f(bxa)$. Thus $M^{\vee\text{op}}$ is again an A bimodule. Note that an A bimodule morphism $M \rightarrow N$ induces an A^{op} module morphism $N^\vee \rightarrow M^\vee$. A *self-dual* A bimodule M is one with an A bimodule isomorphism $\rho : M \rightarrow M^{\vee\text{op}}$ or equivalently an antiisomorphism $\rho^{\text{op}} : M \rightarrow M^\vee$. The latter then gives rise to a non-degenerate bilinear form $\langle -, - \rangle : M \times M \rightarrow k$ by setting $\langle m, m' \rangle = (\rho m)(m')$. However, while the existence of a non-degenerate form $\langle -, - \rangle : M \times M \rightarrow k$ gives rise to a k module monomorphism $\rho : M \rightarrow M^\vee$ by sending m to $\langle m, - \rangle$, it is a bimodule morphism if and only if $\langle abm, m' \rangle = \langle m, bm'a \rangle$ for all $a, b \in A$. In general this induced ρ need not be an epimorphism unless k is a field and M is finite-dimensional. For simplicity we will always tacitly assume that M is free and of finite rank over the coefficient ring k . A sufficient condition that ρ be onto is then that we have a basis $\{m_1, \dots, m_k\}$ such that $\det \langle m_i, m_j \rangle$ is invertible in k , or equivalently, that we have a pair of dual bases $\{m_1, \dots, m_k\}$ and $\{m'_1, \dots, m'_k\}$ for M , i.e., with $\langle m_i, m'_j \rangle = \delta_{ij}$. (The problem when k is not a field can be illustrated by taking $k = \mathbb{Z} = M$. With the bilinear form $\langle m, n \rangle = mn$ the module is actually self dual; $\{1\}$ is a basis and is self dual. However, with the form $\langle m, n \rangle = 2mn$ there no longer exist dual bases – one

would like to take $1^\vee = 1/2$ but that is not in \mathbb{Z} .)

Viewing A as a bimodule over itself, we may in particular consider $A^{\vee\text{op}}$ and call A *quasi self-dual* if there is a canonical isomorphism of graded cohomology modules $H^*(A, A) \cong H^*(A, A^{\vee\text{op}})$. If we have a morphism $B \rightarrow A$ of such algebras then the sequence

$$\begin{aligned} H^*(A, A) \cong H^*(A, A^{\vee\text{op}}) &\rightarrow H^*(B, A^{\vee\text{op}}) \cong H^*(B^{\text{op}}, A^\vee) \\ &\rightarrow H^*(B^{\text{op}}, B^\vee) \cong H^*(B, B^{\vee\text{op}}) \cong H^*(B, B) \end{aligned}$$

exhibits $H^*(A, A)$ as a contravariant functor of A . This principle is not restricted to associative algebras but clearly holds equally well, e.g., for Lie algebras. The direct sum of quasi self-dual algebras is again such. Since we are considering only algebras of finite rank (partly to avoid topological problems) we have $(A_1 \otimes A_2)^\vee = A_1^\vee \otimes A_2^\vee$. This, together with the fact that in general if M_1, M_2 are A_1, A_2 bimodules, respectively, then $H^*(A_1 \otimes A_2, M_1 \otimes M_2) = H^*(A_1, M_1) \otimes H^*(A_2, M_2)$ shows that the category of quasi self-dual algebras is closed under tensor products. It is clearly also closed under direct sums but is not closed under taking quotients, as will be seen.

Suppose now that we have a k module morphism $\phi : A \rightarrow A^\vee$ which for the moment need be neither a monomorphism nor epimorphism. Then we can define a bilinear form $\langle -, - \rangle : A \times A \rightarrow k$ by $\langle a, b \rangle = (\phi a)(b)$, and conversely. The condition that ϕ be an antimorphism from A viewed as an A bimodule to A^{op} viewed as an A^{op} bimodule then is equivalent to having both

$$\langle ac, b \rangle = \langle a, cb \rangle \quad \text{and} \quad \langle ca, b \rangle = \langle a, bc \rangle . \quad (1)$$

If the algebra is unital (which we generally assume) then the form must be symmetric for we have

$$\langle c, b \rangle = \langle 1 \cdot c, b \rangle = \langle 1, cb \rangle = \langle b \cdot 1, c \rangle = \langle bc, 1 \rangle = \langle b, c \rangle .$$

The conditions (1) are equivalent if the form is symmetric, and that in turn will be the case whenever the algebra is commutative, but a non-commutative algebra may still have a symmetric form, e.g. the finite group rings below. If a bilinear form for which the associated k linear mapping $A \rightarrow A^\vee$ is an isomorphism satisfies the conditions (1) then that form will be called *dualizing* and an algebra with a dualizing form will be called *self-dual* since these are precisely the ones with an A bimodule isomorphism $A \rightarrow A^{\vee\text{op}}$; they are in particular quasi self-dual. When A is graded then by ‘‘commutative’’ we will always mean commutative in the graded sense (sometimes called ‘‘supercommutative’’). In that case, if $\deg a = r$, $\deg b = s$ then a ‘symmetric’ form must have $\langle a, b \rangle = (-1)^{rs} \langle b, a \rangle$.

The classical definition of a *Frobenius algebra* A (cf. Nakayam, [10]) is one which is a finite dimensional associative algebra over a field k together with a linear functional $f : A \rightarrow k$ satisfying any of the following conditions which are equivalent in the presence of finite dimensionality: (i) the kernel of f contains no left ideal, (ii) the kernel of f contains no right ideal, (iii) the bilinear ‘‘Frobenius

form" $\langle -, - \rangle: A \times A \rightarrow k$ defined by $\langle a, b \rangle = f(ab)$ is non-degenerate. The form $\langle -, - \rangle$ then has the property that $\langle a, bc \rangle = \langle ab, c \rangle$, and conversely, the existence of a non-degenerate bilinear form with this property implies that A is Frobenius by setting $f(a) = \langle a, 1 \rangle$. A *symmetric Frobenius algebra* is one for which the form is symmetric.

Theorem 2 *A finite dimensional associative algebra over a field is self-dual if and only if it is a symmetric Frobenius algebras. In particular commutative Frobenius algebra, are self-dual.*

PROOF. With the above notation we now have both $\langle a, cb \rangle = \langle ac, b \rangle$ (automatic from the definition) and $\langle a, bc \rangle = \langle bc, a \rangle = \langle b, ca \rangle = \langle ca, b \rangle$, using the symmetry, so the Frobenius form is dualizing. \square

A sufficient condition for symmetry of a Frobenius algebra A is that it possess an involution σ (i.e., antiautomorphism whose square is the identity) preserving the Frobenius form i.e., with $\langle \sigma a, \sigma b \rangle = \langle a, b \rangle$ (or equivalently, if the form is defined by the functional $f: A \rightarrow k$, with $f(\sigma a) = f(a)$). For then we have

$$\begin{aligned} \langle a, bc \rangle &= \langle \sigma a, \sigma(bc) \rangle = \langle \sigma a, \sigma c \cdot \sigma b \rangle \\ &= \langle \sigma a \cdot \sigma c, \sigma b \rangle = \langle \sigma(ca), \sigma b \rangle = \langle ca, b \rangle . \end{aligned}$$

There are important examples of symmetric Frobenius algebras. The group algebra kG of a finite group G is Frobenius: define $\langle a, b \rangle$ to be the coefficient of the identity $1 = 1_G$ in the product ab . This, however, is the same as the coefficient of 1 in ba , so kG is symmetric. In fact, here we do not have to assume that k is field, for if we take as a basis for kG the elements of G then the dual basis consists of their inverses. Thus the group ring kG of a finite group G over any (commutative, associate, unital) ring k is self-dual. The map sending g to g^{-1} , extended linearly, is an involution preserving the form. Another example is the de Rham cohomology ring of a compact manifold \mathcal{M} where, if a, b are cocycles then one sets

$$\langle a, b \rangle = \int_{\mathcal{M}} a \wedge b \quad .$$

We should like to be able to do the same for cohomology with integer coefficients, defining $\langle a, b \rangle$ to be the evaluation of $a \smile b$ on the fundamental cycle, but in addition to the general problem when the coefficient ring is not a field there may now be torsion. In some favorable cases, however, this is still possible

Commutative Frobenius algebras have recently been shown to play an important role in the algebraic treatment and axiomatic foundation of topological quantum field theory since such an algebra determines uniquely (up to isomorphism) a (1+1)-dimensional TQFT. More precisely, the category of commutative Frobenius k algebras is equivalent to the category of symmetric strong monoidal functors from the category of 2-dimensional cobordisms between 1-dimensional manifolds to the category of vector spaces over k .

Closely related to the group algebras of finite commutative groups are the algebras of the form $k[t]/t^{n+1}$ and their tensor products. For let C_q denote the

(multiplicative) cyclic group of order q , let a be a generator, p be a prime, and set $q = p^r$. If k has characteristic p , then $kC_q \cong k[t]/t^q$, the isomorphism being given by $1 - a \mapsto t$. (Since for finite groups G_1 and G_2 we have $k(G_1 \times G_2) = kG_1 \otimes kG_2$, when dealing with commutative groups one need only consider the cyclic case.)

Algebras of the form $k[t]/t^{n+1}$ are Frobenius. For letting u_0, \dots, u_n be the dual basis to $1, t, \dots, t^n$ we have $t \cdot u_i = u_{i-1}, i = 1, \dots, n; t \cdot u_0 = 0$, so we can set $\langle t^i, t^j \rangle = 1$ if $i + j = n$, and set it equal to 0 otherwise. It follows, e.g., that $k[x]/x^r \otimes k[x]/x^s$ is Frobenius for all r and s , hence self-dual. Taking $r = s = 2$ it follows that $k[x, y]/(x^2, y^2)$ is Frobenius. This is free of rank 4 over k , therefore of dimension 4 when k is a field. However, its 3-dimensional quotient $A = k[x, y]/(x^2, y^2, xy)$ is not self-dual, for one can check that $H^1(A, A)$ has dimension 4 but $H^1(A, A^\vee)$ has dimension 3. (Note here that $A = A^{\text{op}}$ and that A^\vee is symmetric.) It follows that $H^*(A, A^{\text{op}})$ is not isomorphic to $H^*(A, A)$ so A is not quasi self-dual, and in particular not Frobenius. This example shows that a quotient of a quasi self-dual algebra need not be quasi self-dual.

That $k[t]/t^{n+1}$ is Frobenius may be viewed as one of the ‘‘favorable cases’’ mentioned above: The cohomology of complex projective n -space $\mathbb{C}P^n$ is isomorphic to $k[t]/t^{n+1}$ where t is the cohomology class of dimension 2 defined by any hyperplane. (Since the dimension is even the algebra is actually commutative in the ordinary sense.) In this case the form mentioned above, evaluation of the cup product on the fundamental cycle, is indeed dualizing.

If G is a group whose order is invertible in k then $H^n(kG, M) = 0$ for all $n > 0$ and all kG modules M (a form of Maschke’s theorem, cf. the next section) but for $k = \mathbb{Z}$ these cohomology groups are generally not trivial although they are all necessarily torsion modules. (Computing the cohomology groups $H^n(\mathbb{Z}C_n, \mathbb{Z})$ is greatly simplified by the fact that \mathbb{Z} , viewed as a trivial $\mathbb{Z}C_n$ module, has a particularly simple projective resolution which is periodic of order 2 and in which every projective module is just $\mathbb{Z}C_n$ itself. This property is shared by all $k[t]/t^{n+1}$. We have for all k

$$\dots k[t]/t^{n+1} \xrightarrow{\partial_i} k[t]/t^{n+1} \xrightarrow{\partial_{i-1}} \dots \xrightarrow{\partial_1} k[t]/t^{n+1} \xrightarrow{\epsilon} k$$

where ϵ is reduction mod t and ∂_i is multiplication by t for i odd and multiplication by t^n for i even.)

The graded Lie structure on $A = kG$ or $A = k[t]/t^{n+1}$ $H^*(A, A)$ is generally not trivial as one can see already by considering the commutator of derivations. In the case of $k[t]/t^{n+1}$, for example, if $D_1t = t^r$ and $D_2t = t^s$ then $D_1D_2 = t_{r+s-1}$, so $[D_1, D_2]t = (s - r)t^{r+s-1}$, which is generally not 0. (We must assume that $1 \leq r, s \leq n$ except, e.g. in characteristic p where for example in $A = k[t]/t^{p^r}$ one can allow $Dt = 1$.) Both $H^*(kG, kG)$ and $H^*(kG, k)$ are rings with multiplications induced by those in the respective coefficient modules kG and k , and the module morphism $kG \rightarrow k$ is in fact a ring morphism; it induces a ring morphism $H^*(kG, kG) \rightarrow H^*(kG, k)$, but the graded Lie part of the Gerstenhaber algebra structure on $H^*(kG, kG)$ is lost.

3 The Lie case

The basic result for Lie algebras is the following

Theorem 3 *Let \mathfrak{g} be a Lie algebra with a bilinear form $\langle -, - \rangle$ for which the associated mapping $\phi : a \rightarrow a^\vee$ is a k linear isomorphism. Then the form is dualizing if and only if it is invariant, i.e., that $\langle [c, a], b \rangle + \langle a, [c, b] \rangle = 0$ for all $a, b, c \in \mathfrak{g}$.*

PROOF. Suppose that $\phi : \mathfrak{g} \rightarrow \mathfrak{g}^\vee$ is a k module isomorphism giving rise to the form $\langle a, b \rangle = (\phi a)(b)$. To be a module morphism we must have $(\phi([c, a]))(b) = [c, \phi a](b)$ for all a, b, c . The left side is $\langle [c, a], b \rangle$; the right (recalling that ϕa is a k module map from \mathfrak{g} to the trivial k module) is $-\langle a, [c, b] \rangle$. \square

In the Lie case a dualizing form need not be symmetric, as we shall see by example. However, we have

Lemma 1 *If a Lie algebra \mathfrak{g} of characteristic $\neq 2$ has a skew dualizing form then \mathfrak{g} is Abelian*

PROOF. Suppose that $\langle -, - \rangle$ is a skew dualizing form on \mathfrak{g} . Then $\langle [c, b], a \rangle = -\langle b, [c, a] \rangle = \langle [c, a], b \rangle$, but the left term equals $\langle c, [b, a] \rangle$ and the right equals $\langle c, [a, b] \rangle$. Assuming that the characteristic is not 2 this implies that $\langle c, [a, b] \rangle = 0$ for all a, b, c ; since the form is non-degenerate one has $[a, b] = 0$ for all a and b , so the Lie algebra is Abelian. \square

The close relationship between associative self-dual and Frobenius algebras no longer holds in the Lie case. A *Frobenius Lie algebra* \mathfrak{g} is one which is finite dimensional over a field and has a functional $f : L \rightarrow k$ such that the skew bilinear *Kirillov form* $\langle a, b \rangle = f([a, b])$ is non-degenerate. The Kirillov form is by definition a coboundary in the Chevalley-Eilenberg theory; a quasi-Frobenius Lie algebra is one with a skew non-degenerate ‘Kirillov’ form which is just a 2-cocycle. By the foregoing the Kirillov form can not be dualizing. Conceivably a Frobenius or quasi-Frobenius Lie algebra could still be self-dual, but relative to some form other than the Kirillov form.

The Killing form of a finite dimensional semisimple real or complex Lie algebra is non-degenerate and invariant so these algebras are self-dual. However, the cohomology of any such algebra with coefficients in any finite dimensional module other than the trivial module vanishes. For an example where this is not the case, let V be a real vector space with an inner product $\langle -, - \rangle_V$, \mathfrak{G} be its orthogonal group and \mathfrak{g} be the Lie algebra of G . Then V is a \mathfrak{g} module and $\langle -, - \rangle_V$ is invariant. Now consider the semi-direct product $\mathfrak{g} \ltimes V$ (a split extension of \mathfrak{g} by V); its Lie multiplication is given by $[(a, v), (b, w)] = ([a, b], [a, w] - [b, v])$. The cohomology of such a Lie algebra with coefficients in itself is generally not zero cf. e.g. [11]. Denoting the Killing form on \mathfrak{g} by $\langle -, - \rangle_K$, the symmetric form on $\mathfrak{g} \ltimes V$ defined by $\langle (a, v), (b, w) \rangle = \langle a, b \rangle_K + \langle v, w \rangle_V$ is non-degenerate and invariant, hence dualizing. For an example of a case where the dualizing form is neither symmetric nor skew, let V be of even dimension with a non-degenerate skew form, \mathfrak{G} be its symplectic

group, and perform the same construction. It should be noted that a finite dimensional real or complex Lie algebra with a non-degenerate invariant form and no one-dimensional ideal is in fact just a semi-direct product of a semisimple Lie algebra with some module over that algebra having no irreducible component of dimension one. This is a consequence of [9, Theorem 3, p.71] and the fact that the cohomology of a semisimple Lie algebra with coefficients in a module whose decomposition into simple components has no component of dimension 1 vanishes in every positive dimension, and in particular, in dimension 2.

4 Separable algebras and relative Hochschild cohomology

A unital k -algebra S is *separable* (over k) if it is projective in the category of S bimodules. Viewing both S and $S \otimes S^{\text{op}}$ as S bimodules, the multiplication map $\mu : S \otimes S^{\text{op}} \rightarrow S$, which is an S bimodule morphism, then has a splitting $\nu : S \rightarrow S \otimes S^{\text{op}}$, i.e., an S bimodule morphism such that $\mu\nu\mu = \mu$. Since $S \otimes S^{\text{op}}$ is a free bimodule of rank 1 over S , the existence of such a splitting is equivalent to separability, since it exhibits S as a direct summand of a free module (for note that $S \otimes S^{\text{op}}$ is a free S bimodule of rank 1 with basis $1 \otimes 1$), which is equivalent to being projective. If we have such a bimodule splitting, for the moment write $\nu(1_S) = \sum x_i \otimes y_i = e_{\text{sep}}$ or simply e . Since $s\nu(1) = \nu(s \cdot 1) = \nu(1 \cdot s) = \nu(1)s$ for all $s \in S$, this e has the remarkable property that

$$\sum s x_i \otimes y_i = \sum x_i \otimes y_i s, \quad \text{all } s \in S.$$

Since $1 = \mu(e) = \sum x_i y_i$ it follows that e is idempotent; it is called the *separability idempotent* of S . Using it we can show that any morphism $f : M \rightarrow N$ of left S modules which *a priori* splits only as a morphism of k modules actually splits as a morphism of left S modules. For if $g : N \rightarrow M$ is a k morphism such that $f g f = f$ then $\bar{g} : N \rightarrow M$ defined by $\bar{g}(n) = \sum_i x_i g(y_i n)$ is a left S module morphism such that $f \bar{g} f = f$. If g is already a left module morphism then \bar{g} will be identical with g , so e projects k module splittings onto S module splittings. The same holds for right modules and also for bimodules by similar arguments. The last is equivalent to separability for μ always has a k splitting: send $s \in S$ to $s \otimes 1$, and this can by hypothesis then be projected onto a bimodule splitting. The one-sided conditions, however, are not strong enough, since if k is a field and S an inseparable extension then a left S module is just a vector space, so the left and right splitting properties both hold. But $S \otimes S$, will then contain a non-trivial radical and the bimodule splitting property will not hold. The term “separable” derives ultimately from the fact that a finite field extension is separable in the classical sense precisely when it is so in that above. (The latter definition is due to M. Auslander and O. Goldman, [1], based on remarkable previous work by Azumaya, [2].)

Using the definition of cohomology by projective resolutions, it follows immediately from the two-sided splitting condition that if S is separable over k

then $H^n(S, M) = 0$ for all $n \geq 1$ and all S bimodules M . The latter is, in fact, another equivalent criterion for separability. By dimension shifting techniques one need in fact only assume that $H^1(A, M) = 0$ for all M and it is this latter that is equivalent to the two-sided splitting condition. The vanishing of cohomology is the basic property of separability that we will need.

It is usually easiest to prove separability by exhibiting the separability idempotent. The algebra of $n \times n$ matrices over k is separable; its separability idempotent is $\sum_i e_{i,1} \otimes e_{1,i}$. (The fixed internal index 1 can be replaced by any other; the separability idempotent is generally not unique.) The group ring kG of a finite group is separable over k if the order of G is invertible in k ; if the order is N then $e = \frac{1}{N} \sum_{g \in G} g \otimes g^{-1}$. (This, in essence, is Maschke's theorem.) Tensor products and finite direct sums of separable algebras are separable. In particular, k is separable over itself and therefore a finite direct sum of copies of k is separable, a fact which we will need. A separable k algebra which is projective as a module over k must be finitely generated; in particular, an algebra which is separable over a field is finite dimensional. (For a readable discussion of separability, cf. [3].)

The use of separability can sometime simplify the computation of Hochschild cohomology. First, call an n -cochain F in $C^n(A, M)$ *normalized* if it vanishes whenever any of its arguments is the unit element of k . These cochains form a subcomplex of the full Hochschild complex whose inclusion into the full Hochschild complex induces an isomorphism of cohomology. Suppose now that S is a k -subalgebra of A , arbitrary except that we will always assume that the unit element of A is contained in S . An S -relative cochain $F \in C^n(A, M)$ is one such that for all $a_1, \dots, a_n \in A$ and $s \in S$ we have

$$\begin{aligned} F(a_1, \dots, a_i s, a_{i+1}, \dots, a_n) &= F(a_1, \dots, a_i, s a_{i+1}, \dots, a_n), & i = 1, \dots, n-1, \\ F(s a_1, \dots, a_n) &= s F(a_1, \dots, a_n), & \text{and} \\ F(a_1, \dots, a_n s) &= F(a_1, \dots, a_n) s. \end{aligned}$$

If, moreover, F is normalized then it must vanish whenever any argument is in S . The relative cochain groups, denoted $C^n(A, S; M)$, also form a subcomplex of the Hochschild complex. The result we need is that when S is a separable algebra over k the inclusion of the complex of S -relative cochains into the full Hochschild complex induces an isomorphism of cohomology. This may be difficult to see from the original Hochschild definition of cohomology but is relatively transparent from the point of view of projective resolutions since separability actually allows one to take tensor products over S rather than k in the bar resolution. Finally, the normalized relative cochain groups $\bar{C}^n(A, S; M)$ form a subcomplex of the relative groups and their inclusion into the full Hochschild cochain complex again induces an isomorphism of cohomology. It is this last subcomplex of normalized S -relative cochains which will be essential in the next section.

5 Poset algebras

Let $\mathcal{P} = \{\dots, i, j, \dots\}$ be a finite poset of order N with no cycles and partial order denoted by \preceq . Extending the partial order to a total order we may assume that $\mathcal{P} = \{1, \dots, n\}$ where the partial order \preceq is compatible with the natural order. Associated to \mathcal{P} is the algebra $A = A(\mathcal{P})$ of upper triangular matrices spanned by the matrix units e_{ij} with $i \preceq j$; these are closed under multiplication. The subalgebra S spanned by the $e_{ii}, i = 1, \dots, N$ has the same unit as A and is isomorphic to a direct sum of copies of k , hence is separable, so we may compute the Hochschild cohomology of A with coefficients in any module, and in particular $H^*(A, A)$ and $H^*(A, A^{\text{op}})$, using S -relative cochains.

Viewing \mathcal{P} as a small category in which $\text{Hom}(i, j)$ consists of a single morphism $i \rightarrow j$ if $i \preceq j$ (the identity morphism when $i = j$) and is empty otherwise, its *nerve* $\Sigma = \Sigma(\mathcal{P})$ is the simplicial complex whose n -simplices σ are the n tuples of composable morphisms $i_0 \rightarrow i_1 \rightarrow \dots \rightarrow i_n$, whose module C_n of n -chains consists of the linear combinations of these, and where $\partial_0\sigma$ just omits the first morphism, $\partial_n\sigma$ omits the last and $\partial_r\sigma, 0 < r < n$ is obtained by replacing $i_{r-1} \rightarrow i_r \rightarrow i_{r+1}$ by $i_{r-1} \rightarrow i_{r+1}$. (The 0-simplices or vertices are just the elements of \mathcal{P} and $\partial_0(i_0 \rightarrow i_1) = i_1, \partial_1(i_0 \rightarrow i_1) = i_0$.) One sets $\partial\sigma = \sum_{i=0}^n (-1)^i \partial_i\sigma$. Denote the simplicial cohomology of $\Sigma = \Sigma(\mathcal{P})$ with coefficients in k by $H^*(\Sigma, k)$. With these notations we can recapitulate the theorem which is the basis of the more general result in [7].

Theorem 4 *There is a canonical isomorphism $H^*(A, A) \cong H^*(\Sigma, k)$.*

Proof. We can compute the left side using S -relative cochains $F \in C^n(A, S; A)$. A cochain is completely determined when its arguments are amongst the e_{ij} . If F is S -relative then

$$F(\dots, e_{ij}, e_{kl}, \dots) = F(\dots, e_{ij}e_{jj}, e_{kl}, \dots) = F(\dots, e_{ij}, e_{jj}e_{kl}, \dots).$$

This vanishes if $j \neq k$ so the only non-zero values of F are those of the form $F(e_{i_0i_1}, e_{i_1i_2}, \dots, e_{i_{n-1}i_n})$, where $i_0 \preceq i_1 \preceq \dots \preceq i_n$. Also, since F is S -relative these values must lie in $e_{i_0i_0}Ae_{i_ni_n}$. The latter is a free module of rank 1 spanned by $e_{i_0i_n}$ so the value is $\lambda e_{i_0i_n}$ for some $\lambda \in k$. Thus F assigns to every $\sigma = (i_0 \rightarrow i_1 \rightarrow \dots \rightarrow i_n)$ an element $\lambda \in k$ and so may be viewed as an element of $C^n(\Sigma, k)$. Conversely, every such simplicial cochain defines a unique S -relative Hochschild cochain. It is easy to check that the simplicial and Hochschild coboundaries then correspond. \square

Any simplicial complex gives rise to a poset whose objects are its faces with the relation ‘‘face of’’. The nerve of this poset is just the barycentric subdivision of the the original simplicial complex and hence has cohomology isomorphic to that of the original.

Corollary 1 *For every finite simplicial complex Σ and coefficient ring k there is a poset k -algebra A such that $H^*(A, A) \cong H^*(\Sigma, k)$. \square*

It is not difficult to see that the isomorphism respects the cup product, so the fact that the cup product in simplicial cohomology is graded commutative is actually a consequence of the more general fact that it is so in $H^*(A, A)$ for every associative algebra A , as first shown in [4].

Let $\{e_{ij}^*\}$ denote the dual basis to $\{e_{ij}\}$. For the operation of A^{op} on the dual bimodule A^\vee a simple computation shows that

$$\begin{aligned} e_{ik}^* e_{lj} &= 0 & \text{if } i \neq l \text{ or } j \not\leq k; & & e_{ik}^* e_{jk} &= e_{jk}^* & \text{if } i \leq j \leq k \\ e_{ij} e_{hl}^* &= 0 & \text{if } j \neq l \text{ or } h \not\leq i; & & e_{ij} e_{hj}^* &= e_{hk}^* & \text{if } h \leq i \leq j \end{aligned}$$

Here for the moment we write simply $e_{ik}^* e_{lj}$ (and not $e_{ik}^* \circ e_{jk}$) since in the foregoing we have just the natural operation of A^{op} on A^\vee .

Theorem 5 *If A is a poset algebra then there is a natural isomorphism*

$$H^*(A, A) \cong H^*(A, A^{\vee \text{op}});$$

poset algebras are quasi self-dual.

Proof. Using S -relative cochains as before we find that $F(e_{i_0 i_1}, e_{i_1 i_2}, \dots, e_{i_{n-1} i_n})$ lies in $e_{i_0 i_0} \circ A^{\vee \text{op}} \circ e_{i_n i_n} = e_{i_n i_n} A^\vee e_{i_0 i_0}$ which is again a free module of rank 1, spanned by $e_{i_0 i_n}^*$. The rest follows as before. \square

Since poset algebras over a fixed ring k are quasi self-dual any algebra morphism between them (or between one and any quasi self-dual algebra) induces a morphism of cohomology. Not all morphisms between poset algebras need be induced by morphisms between the underlying posets but we will not examine here which others are possible. As mentioned, they are probably very restricted.

What has been proven here for poset algebras actually applies to a wider class of algebras called “triangular” in [6]; these are deformations of poset algebras. However, from the deformed algebra one can reconstruct the original poset, and hence the original poset algebra. The cohomologies of the deformed algebra and of the original algebra (each with coefficients in itself) are identical and these are identical with the simplicial cohomology of the nerve of the poset. Thus in the very special case of poset algebras deformation does not alter cohomology. It follows that triangular algebras in the sense of [6] are also quasi self-dual. It seems curious that we may view any space with a finite triangulation as having a “hidden” algebra structure depending on which deformation of its poset algebra we associate to it, but the family of these is already completely determined by the cohomology of the space alone. Perhaps for these spaces the concept of “space” should mean not just the topological object but a pair consisting of that object together with the algebra associated to it.

Classifying self-dual algebras would seem to be a difficult question, particularly since in the associative case this would mean classifying 1+1 topological quantum field theories; the Lie case might be more tractable. Classifying all quasi self-dual algebras would, of course be even more difficult. However, even without classification it might be possible to say when a quasi self-dual algebra

is related in some sense to a geometric object, perhaps closely as with a poset algebra and the nerve of its poset, or in a more extended way as in the relation between symmetric Frobenius algebras and field theories.

6 Deformation of $\mathbb{C}[t]/t^{n+1}$ as a Frobenius algebra.

A natural question concerning each of the structures we have encountered is, What are their deformation theories? In particular, is a deformation of a self-dual algebra again self-dual and a deformation of a Frobenius algebra again Frobenius? This is the case for the commutative Frobenius algebra $A = \mathbb{C}[t]/t^{n+1}$ encountered earlier. The only deformations of this algebra are equivalent to ones of the form $A_{\hbar,p} = \mathbb{C}[t]/(t^{n+1} - \hbar p(t))$ where $p(t)$ is a polynomial of degree $\leq n$ and \hbar is the deformation parameter. (A quotient of a fixed algebra by a varying ideal depending on some parameter \hbar can not always be exhibited as a deformation in the sense of [5] of the quotient obtained when \hbar is set equal to zero since the dimension of that algebra may be different from that for a generic value of \hbar . In a note in preparation I show that most of the difficulties can be resolved with the use of Gröbner bases but in the present case there is certainly no problem since there is a fixed basis for the quotient algebra, namely $\{1, t, t^2, \dots, t^n\}$ independent of the value of \hbar .)

In some sense the simplest Frobenius form on $\mathbb{C}[t]/t^{n+1}$ is that given earlier, in which $\langle t^i, t^j \rangle = 1$ if $i + j = n$ and is 0 otherwise; the form is not unique. For A_{\hbar} , however, one can show that the values of $\langle 1, t^i \rangle$, $i = 0, \dots, n$ completely determine the form. For simplicity, write $\langle t^i, t^j \rangle = c_{i,j}$ and set $p(t) = a_0 + a_1 t + \dots + a_n t^n$. Note that

$$\langle t^i, t^j \rangle = \langle t \cdot t^{i-1}, t^j \rangle = \langle t^{i-1}, t \cdot t^j \rangle = \langle t^{i-1}, t^{j+1} \rangle$$

so the value of $c_{i,j}$ depends only on $i + j$. Then from $\langle t^i, t^n \rangle = \langle t^{i-1}, t^{n+1} \rangle = \langle t^{i-1}, \hbar p(t) \rangle$ we get the recursion

$$c_{i,n} = \hbar(c_{i-1,0}a_0 + c_{i-1,1}a_1 + \dots + c_{i-1,n}a_n).$$

The values of $c_{0,0}, c_{0,1}, \dots, c_{0,n-1}, c_{0,n}$ therefore determine all the $c_{i,j}$. Setting these equal to $0, 0, \dots, 0, 1$, respectively, the matrix $\|c_{i,j}\|$ of the resulting form has 1 everywhere on the main antidiagonal, zero entries above the main antidiagonal and polynomials in \hbar without constant term below the main antidiagonal. Its determinant is therefore $(-1)^{\lfloor \frac{n+1}{2} \rfloor}$, so the resulting form is non-degenerate (and would be with any coefficient ring). The form on A_{\hbar} varies continuously with \hbar and reduces to the original simple form on A when $\hbar = 0$. So the algebra deformation from A to A_{\hbar} has “dragged along” a deformation of the Frobenius form on A to one on A_{\hbar} .

Now suppose that $P(t)$ is a monic polynomial in $\mathbb{C}[t]$ of degree $n + 1$ and that it factors into $P(t) = (t - r_1)^{n_1} (t - r_2)^{n_2} \dots (t - r_k)^{n_k}$. Then by the Chinese

Remainder Theorem,

$$\begin{aligned}\mathbb{C}[t]/P(T) &\cong \mathbb{C}[t]/(t - r_1)^{n_1} \oplus \mathbb{C}[t]/(t - r_2)^{n_2} \cdots \oplus \mathbb{C}[t]/(t - r_k)^{n_k} \\ &\cong \mathbb{C}[t]/(t)^{n_1} \oplus \mathbb{C}[t]/(t)^{n_2} \cdots \oplus \mathbb{C}[t]/(t)^{n_k}.\end{aligned}$$

The structure of $\mathbb{C}[t]/P(T)$ therefore depends only on the partition $\{n_1, n_2, \dots, n_k\}$ of $n + 1$ (where we may assume, without loss of generality, that $n_1 \geq n_2 \geq \dots, n_k$). Since for small non-zero values of \hbar the partition associated with the factorization of $P(t) = t^{n+1} - \hbar p(t)$ is constant, the deformation A_{\hbar} is in fact a jump deformation; all sufficiently small non-zero values of \hbar give isomorphic algebras. There is a natural partial order to partitions of an integer $n + 1$, with $\{n + 1\}$ the largest and $\{1^{n+1}\}$ (meaning 1 repeated $n + 1$ times) the smallest, and this partial order determines a partial order on the possible deformations of $\mathbb{C}[t]/t^{n+1}$, which is itself the highest in the partial order. The lowest is any $\mathbb{C}[t]/(t^{n+1} - a)$ with a any non-zero complex number; these are all isomorphic to a direct sum of $n + 1$ copies of \mathbb{C} .

Referring back to the associated TQFTs, they deform, but the deformations are jumps; one can not “see” any intermediate stages. This raises the questions of whether there is any intrinsic way to parameterize 1+1 dimensional TQFTs without referring to their associated algebras, and whether, unlike those associated with the deformations of $\mathbb{C}[t]/t^{n+1}$, there are any families which do not consist exclusively of jump deformations.

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